## Math 223, Section 101—Homework \#4

due Friday, October 15, 2010 at the beginning of class
Practice problems. Don't turn in these problems. Most of these problems are easy, and most of them have answers in the back of the book, so they can be useful to you to check your understanding of the material.

1. Friedberg, Insel, and Spence, Section 2.1, p. 74, \#1, \#2, \#4, and \#5
2. Friedberg, Insel, and Spence, Section 2.1, p. 75, \#10 and \#12
3. Friedberg, Insel, and Spence, Section 2.2, p. 84, \#1, \#2(a),(c),(d),(f),(g), and \#3
4. Friedberg, Insel, and Spence, Section 2.2, pp. 84-85, \#5(a)-(c),(e),(f) (replace $F$ by $\mathbb{R}$ throughout)
5. Friedberg, Insel, and Spence, Section 2.2, p. 85, \#10
6. Friedberg, Insel, and Spence, Section 2.3, p. 96, \#1 and \#2
7. Friedberg, Insel, and Spence, Section 2.3, pp. 96-97, \#3(a)
8. Friedberg, Insel, and Spence, Section 2.3, p. 97, \#4(a),(c), \#6, and \#7

Homework problems. Write up and hand in solutions to these problems. Please have your solutions to these problems in the correct order on your pages. Write clearly and legibly, in complete sentences; your solutions will be better if you think about how to best phrase your answer before you begin to write it. Please staple the pages together (unstapled solutions will not be accepted).

1. Consider the function $T: P_{3}(\mathbb{R}) \rightarrow P_{4}(\mathbb{R})$ defined by $T(p(x))=(1-x) p(x)-x p(1-x)$. $T$ is a linear transformation (you don't have to prove this).
(a) Find bases for $N(T)$ and $R(T)$.
(b) Compute the nullity and rank of $T$ from your answers to part (a), and verify the Dimension Theorem for $T$.
(Hint: don't be afraid to expand out $a(1-x)^{3}+b(1-x)^{2}+c(1-x)+d$ and so on.)
2. Friedberg, Insel, and Spence, Section 2.2, pp. 84-85, \#2(b),(e), \#4, and \#5(d)
3. Friedberg, Insel, and Spence, Section 2.2, p. 85, \#8, except replace $F^{n}$ by $\mathbb{R}^{n}$
4. Friedberg, Insel, and Spence, Section 2.2, p. 86, \#15(a),(b)
5. (a) Using the matrices $A, B$, and $C$ defined in Friedberg, Insel, and Spence, Section 2.3, p. 96, \#2(b), compute $C A, B C^{t}$, and $B^{t} A$. Which one of these is easy to check using the answers given in the back of the book?
(b) Friedberg, Insel, and Spence, Section 2.3, pp. 96-97, \#3(b). You can use the computation of $[U]_{\beta}^{\gamma}$ from the back of the book without having to justify it (although you should understand why it's correct!).
6. Friedberg, Insel, and Spence, Section 2.3, p. 97, \#11
7. Let $V, W$, and $X$ be vector spaces, and let $T: V \rightarrow W$ and $U: W \rightarrow X$ be linear transformations.
(a) Suppose that $U T$ is onto. Prove that $U$ is also onto.
(b) Suppose that $U T$ is one-to-one. Prove that $T$ is also one-to-one.
(c) Find an example where $U T$ is both one-to-one and onto, but $U$ is not one-to-one and $T$ is not onto.
8. Let $V$ be an $n$-dimensional vector space, and let $\beta$ and $\gamma$ be two ordered bases for $V$. Suppose that $T: V \rightarrow V$ is a linear transformation such that $[T]_{\beta}^{\gamma}$ is the $n \times n$ identity matrix $I_{n}$.
(a) Is $T$ always the same as the identity linear transformation $I_{V}$ ? If so, why? If not, what relationship between $\beta$ and $\gamma$ will cause $T$ to equal $I_{V}$ ?
(b) Prove that no matter what $\beta$ and $\gamma$ are, $T$ is always both one-to-one and onto. (Hint: Theorem 2.5.)

Bonus question. Let $V$ and $W$ be vector spaces, and let $T_{1}, \ldots, T_{n}$ be linear transformations from $V$ to $W$. Then $\left\{T_{1}, \ldots, T_{n}\right\}$ is a subset of the vector space $\mathcal{L}(V, W)$.
(a) Suppose that for each $1 \leq j \leq n$, we know that

$$
N\left(T_{1}\right) \cap \cdots \cap N\left(T_{j-1}\right) \cap N\left(T_{j+1}\right) \cap \cdots \cap N\left(T_{n}\right) \nsubseteq N\left(T_{j}\right)
$$

Prove that $\left\{T_{1}, \ldots, T_{n}\right\}$ is linearly independent, as a subset of $\mathcal{L}(V, W)$.
(b) Suppose that for each $1 \leq j \leq n$, we know that

$$
R\left(T_{j}\right) \nsubseteq \operatorname{span}\left\{R\left(T_{1}\right) \cup \cdots \cup R\left(T_{j-1}\right) \cup R\left(T_{j+1}\right) \cup \cdots \cup R\left(T_{n}\right)\right\} .
$$

Prove that $\left\{T_{1}, \ldots, T_{n}\right\}$ is linearly independent, as a subset of $\mathcal{L}(V, W)$.

