# Math 223, Section 101—Homework \#5 

due Friday, October 22, 2010 at the beginning of class
Practice problems. Don't turn in these problems. Most of these problems are easy, and most of them have answers in the back of the book, so they can be useful to you to check your understanding of the material.

1. Friedberg, Insel, and Spence, Section 2.4, p. 106, \#1 and \#2
2. Friedberg, Insel, and Spence, Section 2.4, p. 107, \#3, \#4, \#5, and \#7
3. Friedberg, Insel, and Spence, Section 2.4, p. 108, \#19
4. Friedberg, Insel, and Spence, Section 2.5, p. 116, \#1, \#2(a),(c), and \#3(a),(c),(e)
5. Friedberg, Insel, and Spence, Section 2.5, pp. 107-8, \#4
6. Friedberg, Insel, and Spence, Section 2.5, p. 108, \#5 and \#6(a),(c)
7. Friedberg, Insel, and Spence, Section 3.1, p. 151, \#1, \#2, and \#3
8. Let $T: V \rightarrow W$ be an isomorphism, where $V$ is finite-dimensional. For any basis $\beta$ of $V$, prove that there exists a unique basis $\gamma$ of $W$ such that $[T]_{\beta}^{\gamma}$ is an $n \times n$ identity matrix for some $n$.

Homework problems. Write up and hand in solutions to these problems. Please have your solutions to these problems in the correct order on your pages. Write clearly and legibly, in complete sentences; your solutions will be better if you think about how to best phrase your answer before you begin to write it. Please staple the pages together (unstapled solutions will not be accepted).

1. Friedberg, Insel, and Spence, Section 2.4, p. 107, \#9 and \#10(a),(b)
2. Friedberg, Insel, and Spence, Section 2.4, p. 108, \#17
3. Friedberg, Insel, and Spence, Section 2.5, p. 116, \#2(b),(d) and \#3(b),(d),(f)
4. Friedberg, Insel, and Spence, Section 2.5, p. 117, \#6(b),(d)
5. Let $\beta=\left\{x^{3}, x^{3}+x^{2}, x^{3}+x^{2}+x, x^{3}+x^{2}+x+1\right\}$ be a basis for $P_{3}(\mathbb{R})$ (you don't have to prove that it's a basis), and let $\gamma=\left\{1, x, x^{2}, x^{3}\right\}$ be the standard ordered basis for $P_{3}(\mathbb{R})$. Let $T: P_{3}(\mathbb{R}) \rightarrow P_{3}(\mathbb{R})$ be the linear transformation given by $T(p(x))=p^{\prime}(x)$.
(a) Write down $[T]_{\gamma}$.
(b) Verify that

$$
\left(\begin{array}{llll}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{array}\right)^{-1}=\left(\begin{array}{rrrr}
0 & 0 & -1 & 1 \\
0 & -1 & 1 & 0 \\
-1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{array}\right) .
$$

(c) Write down $[T]_{\beta}$. Use your answer to write $\left(x^{3}+x^{2}\right)^{\prime}=3 x^{2}+2 x$ as a linear combination of the polynomials in $\beta$.
6. The purpose of this problem is to make sure that we work through some of the potentially confusing notation from the past week. Reminders about the notation can be found in: Example 3 on page 43, the Definitions on pages 80 and 82, the bottom of page 90 (after the Corollary), and Theorem 2.15 on page 93.
(a) Let $\left\{E^{11}, E^{12}, E^{21}, E^{22}\right\}$ be the standard basis for $M_{2 \times 2}(\mathbb{R})$, in that order. Define $\alpha=\left\{L_{E^{11}}, L_{E^{12}}, L_{E^{21}}, L_{E^{22}}\right\}$. Prove that $\alpha$ is a basis for $\mathcal{L}\left(\mathbb{R}^{2}, \mathbb{R}^{2}\right)$.
(b) Let $\alpha$ be the ordered basis for $\mathcal{L}\left(\mathbb{R}^{2}, \mathbb{R}^{2}\right)$ given in part (a), and let $\beta$ be the standard ordered bases for $\mathbb{R}^{2}$. Let $T: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$ be the linear transformation

$$
T((a, b))=(a+b, 2 b)
$$

Write down $[T]_{\alpha}$ and $[T]_{\beta}$. (Hint: they won't have the same shape.)
(c) Let $T$ and $\beta$ be as above. Calculate $\left([T]_{\beta}\right)^{4}$. Use the answer to calculate $T^{4}((0,1))$.
7. (a) Let $W$ be the subspace of $\mathbb{R}^{3}$ given by $W=\{(a, b, 0): a, b \in \mathbb{R}\}$. Find an explicit isomorphism $T: \mathbb{R}^{2} \rightarrow W$ and prove that it is an isomorphism.
(b) Let $V$ and $W$ be any two vector spaces, with $V$ finite-dimensional (but we don't know anything about $W$ ). Prove that one of $V$ and $W$ is isomorphic to a subspace of the other.
8. (a) Let $A$ be the $3 \times 3$ matrix obtained in the following way: starting from $I_{3}$, first subtract twice the first row from the third row; then multiply the second row by 3 ; then switch the first and third rows. Write $A$ as the product of three elementary matrices.
(b) Using your answer to part (a), compute $A^{-1}$.
(c) Let

$$
B=\left(\begin{array}{rrr}
-2 & 0 & 3 \\
0 & 1 & -4 \\
3 & -1 & 0
\end{array}\right),
$$

and let $C$ be the matrix resulting from adding the second column of $B$ to the first column. Calculate $C^{-1}$. Hint: you know what $B^{-1}$ is from recent lectures.

Bonus question. Friedberg, Insel, and Spence, Section 2.6, p. 125, \#9

