Math 223, Section 101—Homework #9
due Friday, November 26, 2010 at the beginning of class

Practice problems. Don't turn in these problems. Most of these problems are easy, and most of them have answers in the back of the book, so they can be useful to you to check your understanding of the material.

1. Friedberg, Insel, and Spence, Section 6.1, p. 337, #6
2. Let $A, B \in M_{n\times n}(\mathbb{C})$. Let $a_1, \dots, a_n$ be the n vectors formed from the rows of A, and let $b_1, \dots, b_n$ be the n vectors formed from the rows of B. Show that the entries of the product AB^* are given by $(AB^*)_{i,j} = \langle a_i, b_j\rangle$ (using the standard inner product on $\mathbb{C}^n$).
3. Let $c_1, \dots, c_n$ be any positive real numbers. Show that $\langle x, y\rangle = \sum_{j=1}^n c_j x_j \overline{y_j}$ defines an inner product on $\mathbb{C}^n$.
4. Friedberg, Insel, and Spence, Section 6.2, pp. 352–3, #1(a),(b),(f),(g)
5. Friedberg, Insel, and Spence, Section 6.2, p. 353, #2(b),(c),(e),(g),(k)
6. Friedberg, Insel, and Spence, Section 6.2, p. 354, #4
7. Friedberg, Insel, and Spence, Section 6.2, p. 354, #19(a),(b). Hint: first find a basis for W, then an orthogonal basis.
8. Let $\{v_1, \dots, v_k\}$ be an orthogonal set in the inner product space V, and let $W = \operatorname{span}\big(\{v_1, \dots, v_k\}\big)$.
   (a) Let $x \in V$, and let u be the orthogonal projection of x on W. Show that $x - u$ is orthogonal to each of $v_1, \dots, v_k$. Conclude that $x - u$ is orthogonal to every vector in W, and so $x - u \in W^\perp$.
   (b) Show that $W^\perp$ is a subspace of V.
9. Friedberg, Insel, and Spence, Section 6.3, p. 365, #1(a),(c)–(g)
10. Friedberg, Insel, and Spence, Section 6.3, pp. 365–6, #2 and #3(a),(c)
11. Friedberg, Insel, and Spence, Section 6.4, pp. 374–5, #1 and #2(a),(e)

Homework problems. Write up and hand in solutions to these problems. Please have your solutions to these problems in the correct order on your pages. Write clearly and legibly, in complete sentences; your solutions will be better if you think about how to best phrase your answer before you begin to write it. Please staple the pages together (unstapled solutions will not be accepted).

1. Let $f(t) = a_n t^n + \cdots + a_2 t^2 + a_1 t + a_0$ be the characteristic polynomial of an $n \times n$ matrix A.
   (a) Show that $a_n = (-1)^n$, that a_{n-1} equals $(-1)^{n-1}\operatorname{tr}(A)$, and that $a_0 = \det(A)$.
   (b) Assume that $f(t)$ splits. Show that the sum of the eigenvalues of A equals $\operatorname{tr}(A)$ and that the product of the eigenvalues of A equals $\det(A)$. (Here, if an eigenvalue of A has multiplicity m, then we add it to the sum m times, and multiply it into the product m times.) Hint: what do you know about the relationship between the sum and product of roots of a polynomial and the coefficients of the polynomial?
2. Friedberg, Insel, and Spence, Section 6.2, p. 353, #2(f),(h)

(continued on next page)

3. Define an inner product on $P(\mathbb{R})$ by the formula $\langle f(x), g(x)\rangle = \int_0^1 f(t)g(t)\,dt$.
   (a) Use the Gram–Schmidt process on the linearly independent set $R = \{1, x, x^2\}$ to produce an orthogonal basis for $P_2(\mathbb{R})$.
   (b) Use the Gram–Schmidt process on the linearly independent set $S = \{x^2, x, 1\}$ to produce an orthogonal basis for $P_2(\mathbb{R})$.
4. Friedberg, Insel, and Spence, Section 6.2, pp. 354–5, #6 and #13(a)–(c). In both problems, you may assume that V is finite-dimensional.
5. (a) Friedberg, Insel, and Spence, Section 6.3, p. 366, #3(b)
   (b) Define a linear operator T on $M_{2\times 2}(\mathbb{C})$ by $T(A) = A\left(\begin{smallmatrix} i & 3 \\ 2-i & 0 \end{smallmatrix}\right)$. Calculate $T^*(C)$, where $C = \left(\begin{smallmatrix} -3 & 4+3i \\ 2i & 1-5i \end{smallmatrix}\right)$. (Hint: calculate $[T]_\beta$ for a convenient ordered basis β for $M_{2\times 2}(\mathbb{C})$, and apply Theorem 6.10.)
6. Friedberg, Insel, and Spence, Section 6.3, p. 366, #12. You may assume that V is finite-dimensional.
7. Friedberg, Insel, and Spence, Section 6.4, p. 375, #2(b),(d),(f)
8. Friedberg, Insel, and Spence, Section 6.4, p. 376, #9

Bonus question. Let V and W are finite-dimensional inner product spaces (both real or both complex). We would like to understand when a linear transformation $T\colon V \to W$ exists such that

$$\langle T(x), T(y)\rangle = \langle x, y\rangle \text{ for all } x, y \in V. \qquad (*)$$

1. Show that there exists a linear transformation $T\colon V \to W$ satisfying $(*)$ if and only if $\dim(V) \leq \dim(W)$.
2. If $\dim(V) = \dim(W)$, show that any linear transformation $T\colon V \to W$ satisfying $(*)$ is in fact an isomorphism.