Math 223, Section 201
Homework #9
due Wednesday, April 3, 2002 at the beginning of class

Warm-Up Questions—do not hand in

I. Lay, Section 5.2, p. 312, #19

II. Suppose that A, B, C are $n \times n$ matrices such that A and B are similar to each other and B and C are similar to each other. Prove that A and C are similar to each other.

III. How do the eigenvalues and eigenvectors of A^k relate to those of A?

IV. Suppose that the $n \times n$ matrix A has one eigenvalue k with multiplicity n, i.e., that the characteristic polynomial of A is $(k - \lambda)^n$. If A is not already diagonal, prove that A is not diagonalizable.

V. Find two matrices with the same characteristic equation (i.e., with the same eigenvalues including multiplicity) that are not similar to each other.

April 3's quiz will be one of these five warm-up questions.

Homework Questions—hand these in

I. Diagonalize the matrix $A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}$, that is, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

II. Suppose that A is an $n \times n$ matrix such that A^3 is similar to A. Prove that 1, 0, and -1 are the only possible eigenvalues for A.

III. Define a sequence h_n of numbers recursively by $h_0 = 1$, $h_1 = 0$, $h_2 = 6$, and

$$h_{n+3} = 4h_{n+2} - h_{n+1} - 6h_n \qquad (n \geq 0), \tag{1}$$

so that $h_3 = 4 \cdot 6 - 0 - 6 \cdot 1 = 18$, $h_4 = 4 \cdot 18 - 6 - 6 \cdot 0 = 66$, and so on.

(a) Find a formula for h_n that does not involve other terms h_m in the sequence (like we did in class for the Lucas numbers).

(b) What value does h_{n+1}/h_n approach as n gets larger and larger? (The justification for your answer should be convincing, but not necessarily rigorous—this isn't an analysis class after all.)

(c) The sequence could have been defined with any initial values $h_0 = a$, $h_1 = b$, $h_2 = c$ and the same recursive rule (1). Find values for a, b, c that would have made h_{n+1}/h_n approach a different value than the one you found in part (b) above. (In particular, we would need h_n never to equal 0 for this to make sense.)

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IV. For any $n \times n$ matrix M and any polynomial $p(t) = c_0 + c_1 t + \cdots + c_k t^k$, define $p(M)$ to be the matrix $c_0 I_n + c_1 M + c_2 M^2 + \cdots + c_k M^k$.

(a) If $A = PBP^{-1}$ for some $n \times n$ matrices A, P, B and $p(t)$ is any polynomial, prove that $p(A) = Pp(B)P^{-1}$.

(b) If $D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$, prove that $p(D) = \begin{bmatrix} p(d_1) & 0 & \cdots & 0 \\ 0 & p(d_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p(d_n) \end{bmatrix}$.

(c) Let A be an $n \times n$ diagonalizable matrix, and let $p(t)$ be the characteristic polynomial of A. Prove that $p(A)$ is the zero matrix.

V. (a) Prove that $B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 5 \\ -4 \\ -1 \end{bmatrix} \right\}$ is an orthogonal basis for $\mathbb{R}^3$, and find the B-coordinate vector of $\begin{bmatrix} -3 \\ 10 \\ 8 \end{bmatrix}$.

(b) Find an orthogonal matrix whose first column is $\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{14} \\ 5/\sqrt{42} \end{bmatrix}$.

VI. Let B be an $m \times n$ matrix with $m < n$, and let $C = B^T B$. Prove that there exists an orthonormal basis for $\mathbb{R}^n$ that includes a vector in $\operatorname{Nul} C$. (Hint: How large can $\operatorname{rank} C$ be?)