Math 308, Section 101
Final Exam
December 16, 2004
Duration: 150 minutes

Name: ______________________ Student Number: ______________________

Do not open this test until instructed to do so! This exam should have 16 pages, including this cover sheet. No textbooks, notes, calculators, or other aids are allowed. Turn off any cell phones, pagers, etc. that could make noise during the exam. You must remain in this room until you have finished the exam.

All your solutions must be written clearly and understandably. Use complete sentences and explain why your mathematical statements are relevant to the problem. You should always write enough to demonstrate that you're not just guessing the answer. Use the backs of the pages if necessary. You will find some of the questions quite easy; try to solve these first. Good luck!

Be aware of these UBC rules governing examinations:

(a) Each candidate must be prepared to produce, upon request, a Library/AMS card for identification.
(b) Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
(c) No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
(d) Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
- Having at the place of writing any books, papers or memoranda, calculators, computers, audio or video cassette players or other memory aid devices, other than those authorized by the examiners.
- Speaking or communicating with other candidates.
- Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.

(e) Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.

Problem	**Out of**	**Score**
1	4	
2	4	
3	4	
4	4	

Problem	**Out of**	**Score**
5	5	
6	4	
7	4	
8	4	

Problem	**Out of**	**Score**
9	4	
10	4	
11	5	
12	4	
Total	50	

These facts are here for your reference. This is not a complete list; you may quote any facts we have covered, as long as you make it clear what fact you are using. You are not required to quote them by theorem number, lemma number, and so on (but you may if you wish to). **Please do not remove these pages from the test booklet.**

Neutral geometry

The three axioms concerning isometries are as follows.

Axiom 6: For any two points P and Q, there exists an isometry f with $f(P) = Q$.

Axiom 7: For any three points P, Q, and R satisfying $|PQ| = |PR|$, there exists an isometry f with $f(P) = P$ and $f(Q) = R$.

Axiom 8: For any line ℓ, there is an isometry that fixes every point on ℓ and does not fix any other points.

Fact. Angles forming a straight line add to $180°$, and vertical angles are always congruent.

The **SSS**, **SAS**, and **ASA** triangle congruence theorems hold.

Fact. If $\triangle ABC \equiv \triangle A'B'C'$, then the corresponding sides are congruent (for example, $|AB| = |A'B'|$), as are the corresponding angles (for example, $\angle BAC = \angle B'A'C'$).

Pons Asinorum. The base angles of an isosceles triangle are equal: if $|AB| = |AC|$, then $\angle ABC = \angle ACB$. [Note: the converse is also true: if $\angle ABC = \angle ACB$, then $|AB| = |AC|$.]

Fact. Given a ray AB and a positive number x, there is exactly one point C on the ray AB such that $|AC| = x$.

Fact. Suppose that ℓ_1 and ℓ_2 are both perpendicular to some other line ℓ. If ℓ_1 and ℓ_2 have a point in common, then $\ell_1 = \ell_2$.

Euclidean geometry only

Theorem 1.4.1. Let P be a point not on the line ℓ, and let Q be a point on ℓ such that PQ is perpendicular to ℓ. Let ℓ_2 be the line through P that is parallel to ℓ. Then ℓ_2 is perpendicular to PQ.

Corollary 1.4.4. Suppose a line ℓ intersects two distinct lines ℓ_1 and ℓ_2 so that the opposite interior angles are equal. Then ℓ_1 and ℓ_2 are parallel. [Note 1: In particular, if ℓ is perpendicular to both ℓ_1 and ℓ_2, then ℓ_1 and ℓ_2 are parallel.] [Note 2: The converse is also true: if ℓ_1 and ℓ_2 are parallel, then the opposite interior angles formed by ℓ are equal.]

Star Trek Lemma. The measure of an inscribed angle equals half of the angular measure of the arc it subtends. [Note: the angular measure of an arc PQ on a circle centered at O is equal to the measure of the central angle $\angle POQ$.]

Bow Tie Lemma. Let A, A', B, and C lie on a circle, and suppose that $\angle BAC$ and $\angle BA'C$ subtend the same arc. Then $\angle BAC = \angle BA'C$.

Theorem 1.7.1. Let B' and C' be on the sides AB and AC, respectively, of $\triangle ABC$. Then $B'C'$ is parallel to BC if and only if $|AB'|/|AB| = |AC'|/|AC|$.

Corollary 1.7.4. If $\triangle ABC \sim \triangle A'B'C'$, then $|A'B'|/|AB| = |A'C'|/|AC| = |B'C'|/|BC|$. [Note: the converse is also true: if $|A'B'|/|AB| = |A'C'|/|AC| = |B'C'|/|BC|$, then $\triangle ABC \sim \triangle A'B'C'$.]

SAS for Similarity. Let $\triangle ABC$ and $\triangle A'B'C'$ be two triangles such that $\angle BAC = \angle B'A'C'$ and $|A'B'|/|AB| = |A'C'|/|AC|$. Then $\triangle ABC \sim \triangle A'B'C'$.

Euclidean geometry only (continued)

Power of the Point Theorem. Let P be any point and $\mathcal{C}$ any circle, and let ℓ_1 and ℓ_2 be two lines through P. Suppose that ℓ_1 intersects $\mathcal{C}$ in the two points Q and Q', and ℓ_2 intersects $\mathcal{C}$ in the two points R and R'. Then $|PQ||PQ'| = |PR||PR'|$.

Theorem 1.9.1. Let AA', BB', and CC' be the three medians of the triangle $\triangle ABC$. Let G be the centroid of $\triangle ABC$. Then $|AG| = 2|A'G|$, $|BG| = 2|B'G|$, and $|CG| = 2|C'G|$.

Heron's Formula. Let s be the semiperimeter of $\triangle ABC$. Then the area of $\triangle ABC$ is given by the formula $|\triangle ABC| = \sqrt{s(s-a)(s-b)(s-c)}$. [Note: a, b, and c are the lengths of the sides of $\triangle ABC$, and $s = \frac{1}{2}(a+b+c)$.]

Fact. The area of a triangle $\triangle ABC$ is given by the formula $|\triangle ABC| = \frac{1}{2}ab\sin C$.

Law of Cosines. For any triangle $\triangle ABC$, we have $c^2 = a^2 + b^2 - 2ab\cos C$.

Extended Law of Sines. Let R be the circumradius of $\triangle ABC$. Then $a/\sin A = b/\sin B = c/\sin C = 2R$. [Note: the circumradius of a triangle is the radius of the circle that can be circumscribed around it.]

Theorem 1.10.2. Let r be the inradius of $\triangle ABC$, and let s be the semiperimeter of $\triangle ABC$. Then the area of $\triangle ABC$ is given by the formula $|\triangle ABC| = rs$. [Note: the inradius of a triangle is the radius of the circle that can be inscribed inside the triangle.]

Fact about cubics. Let $P(x) = ax^3 + bx^2 + cx + d$ be an integer polynomial. Suppose that $P(r) \neq 0$ for every number r of the form $r = \pm m/n$, where m divides d and n divides a. Then $P(x)$ is irreducible over the integers.

Theorem 3.6.4. Suppose a length $x > 0$ is the root of an irreducible polynomial of degree n. If n is not a power of 2, then x is not constructible.

Theorem 3.6.5. Let p be an odd prime. For any integer $r \geq 2$, it is impossible to construct a regular p^r-gon. Furthermore, a regular p-gon is constructible if and only if p is of the form $2^{2^k} + 1$. [Note: these special primes p are called Fermat primes.]

Hyperbolic geometry only

Theorem 6.3.1. If two triangles are similar, then they are automatically congruent.

Theorem 6.4.2. Two ultraparallel lines have a common perpendicular.

Some examples of **isometries of the Poincaré half-plane:**

- horizontal translations
- dilations (scaling by a positive real number)
- reflection in the y-axis
- inversion in the unit circle
- $z \mapsto -1/z$
- fractional linear transformations: if $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is in $SL_2(\mathbb{R})$, then $T_\gamma(z) = \dfrac{az+b}{cz+d}$
- any composition of these isometries

Also, every isometry of the Poincaré half-plane can be written as either $T_\gamma(z)$ or $T_\gamma(-\bar{z})$ for some matrix γ in $SL_2(\mathbb{R})$.