Math 308, Section 101
Study Questions for Final Exam
(Thursday, December 16, 2004 at 8:30 AM in BUCH A102)

Besides the questions listed below, remember also to review the problems from the Homeworks and the Recommended Exercises given out with the regular reading assignments.

NEUTRAL GEOMETRY

I. Given a triangle $\triangle AEF$, let M be the midpoint of the segment EF. Choose a point D on the ray AM such that $|AD| = 2|AM|$. Prove that $\triangle DFE \equiv \triangle AEF$.

Define the *defect* of a triangle $\triangle ABC$, written $\delta(\triangle ABC)$, to be $\delta(\triangle ABC) = 180° - \angle A - \angle B - \angle C$. Notice that by the Saccheri-Legendre Theorem, the defect of a triangle is always a nonnegative number of degrees.

II. Suppose that D, E, and F are points on the sides BC, AC, and AB of $\triangle ABC$, respectively. Prove that

$$\delta(\triangle ABC) = \delta(\triangle AEF) + \delta(\triangle BDF) + \delta(\triangle CDE) + \delta(\triangle DEF).$$

III. Legendre gave the following incorrect argument when he was trying to prove that the Parallel Postulate follows from the neutral geometry axioms. Find the mistake in the following "proof" that the sum of the angles of every triangle must equal $180°$ (a fact which would indeed imply the Parallel Postulate, if proven correctly).

First we claim that for any triangle $\triangle AEF$, there exists a triangle $\triangle ABC$ whose defect is at least twice as big, that is, $\delta(\triangle ABC) \geq 2\delta(\triangle AEF)$. To see this, let M be the midpoint of the segment EF, and choose a point D on the ray AM such that $|AD| = 2|AM|$; we know that $\triangle DFE \equiv \triangle AEF$. In particular, $\delta(\triangle DFE) = \delta(\triangle AEF)$. Now draw any line through D that intersects the rays AF and AE; call the points of intersection B and C, respectively. We then have

$$\begin{aligned}\delta(\triangle ABC) &= \delta(\triangle AEF) + \delta(\triangle BDF) + \delta(\triangle CDE) + \delta(\triangle DEF)\\ &\geq \delta(\triangle AEF) + 0 + 0 + \delta(\triangle DEF) = 2\delta(\triangle AEF).\end{aligned}$$

Now suppose, for the sake of contradiction, that a triangle $\triangle AEF$ existed with an angle sum less than $180°$, so that $\delta(\triangle AEF) > 0$. We can find a sequence of triangles whose defects are at least $2\delta(\triangle AEF)$, $4\delta(\triangle AEF)$, $8\delta(\triangle AEF)$, and so on. Eventually, the numbers $2^k\delta(\triangle AEF)$ are bigger than $180°$, which is absurd, since the largest possible defect of a triangle is $180°$. Therefore no triangle with positive defect exists, that is, every triangle has angle sum exactly $180°$.

(Remember, this proof is incorrect—what is the mistake?)

EUCLIDEAN GEOMETRY

I. Baragar, p. 139, #7.11

II. In $\triangle ABC$, trisect each side. That is, choose points A' and A'' between B and C such that $|BA'| = |A'A''| = |A''C|$; choose points B' and B'' between C and A such that $|CB'| = |B'B''| = |B''A|$; and choose points C' and C'' between A and B such that $|AC'| = |C'C''| = |C''B|$. Let P be the intersection of the segments $C'A''$ and $A'B''$. Prove that $\triangle C'PB'' \equiv \triangle A''PA'$.

III. Let P and Q be two points on a circle and A a point outside the circle, and let the rays AP and AQ intersect the circle again in the points P' and Q', respectively. Suppose that P is the midpoint of the segment AP' and that Q is the midpoint of the segment AQ'. Prove that $\angle PQP' = \angle QPQ'$.

IV. Given a triangle $\triangle ABC$, let D be a point on BC, E a point on AC, and F a point on AB. Suppose that $|AE| = 3$, $|AF| = 4$, $|BD| = 6$, $|BF| = 8$, and $|CD| = 9$.
  (a) What are the possible values of $|CE|$?
  (b) If the lines AD, BE, and CF all intersect in a single point, what is the value of $|CE|$?

V. Baragar, p. 110, #5.13

VI. Name all (three-dimensional) regular polyhedra, sometimes called "Platonic solids". Which ones are dual to each other? Explain.

VII. How many edges, vertices, triangular faces, and pentagonal faces are there on the snub dodecahedron, which is represented by (3,3,3,3,5)? What about its dual?

HYPERBOLIC GEOMETRY

I. For this question, "draw" means "draw, with reasonable accuracy, a picture using the *Poincaré half-plane model* of the hyperbolic plane". For this question, do not allow any of your hyperbolic lines to look like vertical Euclidean lines.
  (a) Draw three lines $\mathcal{L}$, $\mathcal{M}$, and $\mathcal{N}$ such that $\mathcal{L}$ and $\mathcal{M}$ are parallel to each other, $\mathcal{L}$ and $\mathcal{N}$ are ultraparallel to each other, and $\mathcal{M}$ and $\mathcal{N}$ intersect each other. (Be sure to label the lines correctly.)
  (b) Draw a quadrilateral $ABCD$ such that each of its four angles is less than 30°. (Be sure that all four vertices really are points in the hyperbolic plane.)
  (c) Draw a line $\mathcal{L}$ through a point P, and draw the image of $\mathcal{L}$ under the isometry (of the hyperbolic plane) that is a rotation around P through a 120° angle. Mark the 120° angle in your drawing, and label the line $\mathcal{L}$ to distinguish it from its image line.

II. Suppose that in the Poincaré half-plane, the line AB is an arc of a semicircle with one endpoint at the real number x. Prove that the horizontal translation $z \mapsto z - x$ followed by inversion in the unit circle is an isometry of the Poincaré half-plane that sends the line AB to a vertical line.

III. Find the distance between $15i$ and $-7 + 8i$ in the Poincaré half-plane.

IV. Using the definitions of the hyperbolic trigonometry functions, prove that $\cosh^2 x - \sinh^2 x = 1$.

V. Let $\triangle ABC$ be a right triangle with right angle $\angle C$. Suppose that $|CB| = \ln 5$ and $|CA| = \ln 13$. Verify, using the Hyperbolic Pythagorean Theorem $\cosh c = \cosh a \cosh b$, that $|AB| = \ln(17 + 12\sqrt{2})$.