Math 308, Section 101 Solutions to Study Questions for Final Exam (Thursday, December 16, 2004)

NEUTRAL GEOMETRY

I. Given a triangle $\triangle AEF$, let M be the midpoint of the segment EF. Choose a point D on the ray AM such that |AD| = 2|AM|. Prove that $\triangle DFE \equiv \triangle AEF$.

Note that since M is between A and D, we have |AM| + |MD| = |AD| = 2|AM|, and so |AM| = |MD|. We also know that |FM| = |ME| since M is the midpoint of EF. Finally, the vertical angles $\angle AMF$ and $\angle DME$ are certainly congruent. Therefore, by SAS, $\triangle AMF \equiv \triangle DME$; in particular, |AF| = |DE|. A similar argument shows that $\triangle AME \equiv \triangle DMF$ and hence |AE| = |DF|. Finally, |EF| = |FE| trivially. We now know enough to conclude from SSS that $\triangle DFE \equiv \triangle AEF$.

Define the defect of a triangle $\triangle ABC$, written $\delta(\triangle ABC)$, to be $\delta(\triangle ABC) = 180^{\circ} - \angle A - \angle B - \angle C$. Notice that by the Saccheri-Legendre Theorem, the defect of a triangle is always a nonnegative number of degrees.

II. Suppose that D, E, and F are points on the sides BC, AC, and AB of $\triangle ABC$, respectively. Prove that

$$\delta(\triangle ABC) = \delta(\triangle AEF) + \delta(\triangle BDF) + \delta(\triangle CDE) + \delta(\triangle DEF).$$

The right-hand side is

$$\begin{split} \delta(\triangle AEF) + \delta(\triangle BDF) + \delta(\triangle CDE) + \delta(\triangle DEF) \\ &= (180^{\circ} - \angle AEF - \angle AFE - \angle FAE) + (180^{\circ} - \angle BDF - \angle BFD - \angle DBF) \\ &+ (180^{\circ} - \angle CDE - \angle CED - \angle DCE) + (180^{\circ} - \angle DEF - \angle EFD - \angle FDE) \\ &= 720^{\circ} - \angle FAE - \angle DBF - \angle DCE - (\angle AEF + \angle DEF + \angle CED) \\ &- (\angle CDE + \angle FDE + \angle BDF) - (\angle BFD + \angle EFD + \angle AFE) \\ &= 720^{\circ} - \angle FAE - \angle DBF - \angle DCE - 180^{\circ} - 180^{\circ} \\ &= 180^{\circ} - \angle BAC - \angle CBA - \angle BCA = \delta(\triangle ABC). \end{split}$$

All we used was that angles that together form a straight line add up to 180° .

III. Legendre gave the following incorrect argument when he was trying to prove that the Parallel Postulate follows from the neutral geometry axioms. Find the mistake in the following "proof" that the sum of the angles of every triangle must equal 180° (a fact which would indeed imply the Parallel Postulate, if proven correctly).

First we claim that for any triangle $\triangle AEF$, there exists a triangle $\triangle ABC$ whose defect is at least twice as big, that is, $\delta(\triangle ABC) \ge 2\delta(\triangle AEF)$. To see this, let M be the midpoint of the segment EF, and choose a point D on the ray AM such that |AD| = 2|AM|; we know that $\triangle DFE \equiv \triangle AEF$. In particular, $\delta(\triangle DFE) = \delta(\triangle AEF)$. Now draw any line through D that intersects the rays AF and AE; call the points of intersection B and C, respectively. We then have

$$\begin{split} \delta(\triangle ABC) &= \delta(\triangle AEF) + \delta(\triangle BDF) + \delta(\triangle CDE) + \delta(\triangle DEF) \\ &\geq \delta(\triangle AEF) + 0 + 0 + \delta(\triangle DEF) = 2\delta(\triangle AEF). \end{split}$$

Now suppose, for the sake of contradiction, that a triangle $\triangle AEF$ existed with an angle sum less than 180°, so that $\delta(\triangle AEF) > 0$. We can find a sequence of triangles whose defects are at least $2\delta(\triangle AEF)$, $4\delta(\triangle AEF)$, $8\delta(\triangle AEF)$, and so on. Eventually, the numbers $2^k\delta(\triangle AEF)$ are bigger than 180°, which is absurd, since the largest possible defect of a triangle is 180°. Therefore no triangle with positive defect exists, that is, every triangle has angle sum exactly 180°.

(Remember, this proof is incorrect—what is the mistake?)

The sole problem with the proof is the following statement: "Now draw any line through D that intersects the rays AF and AE". In hyperbolic geometry (and hence in neutral geometry), it is quite possible for a point D to lie inside an angle $\angle EAF$, yet no line exists that intersects both rays AF and AE. It just goes to show how misleading our Euclidean geometry intuition can be when studying hyperbolic geometry. (At least we have an advantage over Legendre: we know that hyperbolic geometry actually exists!)

EUCLIDEAN GEOMETRY

IV. (Baragar, p. 139, #7.11) Let P be a point inside a circle centered at O. Let T be a point where the perpendicular to PO intersects the circle. Let P' be the point where the tangent to the circle at T intersects OP. Show that P' is the image of P under inversion in the circle.

It would be clearer to say "Let T be a point where the perpendicular to PO at P intersects the circle." The outline of the solution is as follows: prove that the right triangles $\triangle OPT$ and $\triangle TPP'$ are similar, using the fact that $\angle OTP + \angle PTP' = 90^{\circ}$. Conclude that |PO|/|OT| =|TO|/|OP'|, which is the equality needed to show that P' is the image of P under inversion in the circle (since P' is already on the ray OP where that image needs to be).

V. In $\triangle ABC$, trisect each side. That is, choose points A' and A" between B and C such that |BA'| = |A'A''| = |A''C|; choose points B' and B" between C and A such that |CB'| = |B'B''| = |B''A|; and choose points C' and C" between A and B such that |AC'| = |C'C''| = |C''B|. Let P be the intersection of the segments C'A" and A'B". Prove that $\triangle C'PB'' \equiv \triangle A''PA'$.

Note that |AC'|/|AB| = 1/3 = |AB''|/|AC|, since we have trisected the various sides. It follows that $\triangle AC'B'' \sim \triangle ABC$ and that the segments C'B'' and BC are parallel. Therefore the opposite interior angles $\angle C'B''P$ and $\angle A''A'P$ are congruent, as are $\angle B''C'P$ and $\angle A'A''P$. We also have that |C'B''|/|BC| = 1/3, again by the similarity of $\triangle AC'B''$ and $\triangle ABC$. But we also have |A'A''|/|BC| = 1/3, and so |C'B''| = |A''A'|. Therefore by ASA, $\triangle C'PB'' \equiv \triangle A''PA'$. [Remark: P is actually the centroid of $\triangle ABC!$]

VI. Let P and Q be two points on a circle and A a point outside the circle, and let the rays AP and AQ intersect the circle again in the points P' and Q', respectively. Suppose that P is the midpoint of the segment AP' and that Q is the midpoint of the segment AQ'. Prove that $\angle PQP' = \angle QPQ'$.

First we prove: if W, X, Y, Z are points on a circle centered at O, and |WX| = |YZ|, then the angular measures of the arcs WX and YZ are the same. To see this, note that $\triangle OWX \equiv \triangle OYZ$ by the SSS theorem, since |OW| = |OY| and |OX| = |OZ| because they are all radii of the same circle, and we are given |WX| = |YZ|. In particular, the central angles $\angle WOX$ and $\angle YOZ$ are congruent, which means that the angular measures of the arcs WX and YZ are the same. Now, using the power of the point theorem, we know that |AP||AP'| = |AQ||AQ'|. But |AP| = |PP'| and |AP'| = 2|PP'| since P is the midpoint of the segment AP', and similarly |AQ| = |QQ'| and |AQ'| = 2|QQ'|; therefore we conclude that $|PP'| \cdot 2|PP'| = |QQ'| \cdot 2|QQ'|$, which implies that |PP'| = |QQ'|. From the fact proved in the previous paragraph, we see that the angular measures of the arcs PP' and QQ' are the same. But by the Star Trek Lemma, $\angle PQP'$ is half the measure of the arc PP', while $\angle QPQ'$ is half the measure of the arc QQ'; therefore $\angle PQP' = \angle QPQ'$ as desired.

VII. Given a triangle $\triangle ABC$, let D be a point on BC, E a point on AC, and F a point on AB. Suppose that |AE| = 3, |AF| = 4, |BD| = 6, |BF| = 8, and |CD| = 9.

- (a) What are the possible values of |CE|?
- (b) If the lines AD, BE, and CF all intersect in a single point, what is the value of |CE|?
- (a) We have |AB| = |AF| + |FB| = 12 and |BC| = |BD| + |DC| = 15, and also |AC| = |AE| + |CE| = 3 + |CE|. By the triangle inequality, we must have $3 + |CE| = |AC| \le |AB| + |BC| = 12 + 15 = 27$, or in other words $|CE| \le 24$. We must also have $15 = |BC| \le |BA| + |AC| = 12 + (3 + |CE|) = 15 + |CE|$, which in this case only gives us the obvious inequality $|CE| \ge 0$. The triangle inequality $|AB| \le |AC| + |BC|$ is equally unhelpful in this case (we obtain $|CE| \ge -6$). Therefore the only restriction is that $0 \le |CE| \le 24$; and if we want to restrict to non-degenerate triangles (those in which the three vertices don't all lie on a line), then we need 0 < |CE| < 24 with strict inequality.
- (b) By Ceva's theorem,

$$\frac{|AF|}{|FB|}\frac{|BD|}{|DC|}\frac{|CE|}{|EA|} = 1,$$

or in other words, $|CE| = |FB||DC||EA|/|AF||BD| = 8 \cdot 9 \cdot 3/(4 \cdot 6) = 9.$

VIII. (Baragar, p. 110, #5.13) Find formulas for F, V, and E in terms of m and n for a Platonic solid with m n-gons at each vertex.

Since each face has n edges and each edge adjoins two faces, we have 2E = nF. Since each face has n vertices and each vertex belongs to m faces, we also have mV = nF. Using this, the Euler formula becomes 2 = F + nF/m - nF/2, which implies that F = 4m/(2m+2n-mn). The first two equations then show that E = 2mn/(2m+2n-mn) and V = 4n/(2m+2n-mn). (Notice that if $mn \ge 2m + 2n$, then the denominator is either zero or negative, and hence no such Platonic solid exists.)

IX. Name all (three-dimensional) regular polyhedra, sometimes called "Platonic solids". Which ones are dual to each other? Explain.

The answer is in Chapter 5 of the textbook. It is important to understand how to form the dual to a convex polyhedron and why this makes the cube and the octahedron, for example, each other's duals.

X. How many edges, vertices, triangular faces, and pentagonal faces are there on the snub dodecahedron, which is represented by (3,3,3,3,5)? What about its dual?

The textbook (Baragar, page 109) tells us that there are $F_5 = 12$ pentagonal faces on the snub dodecahedron, and also gives us the equations $V = 5F_5 = 3F_3/4$ and $E = (3F_3+5F_5)/2$.

We quickly find V = 60, $F_3 = 80$, and E = 150, so the snub dodecahedron has 60 vertices, 150 edges, and 80 triangular faces for a total of 92 faces.

The dual of a convex polyhedron is formed by having one vertex for every face of the original polyhedron, and one face for every vertex. In this case, the dual polyhedron will have 92 vertices and 60 faces (and 150 edges again, since the equation V - E + F = 2 still holds). Since each vertex of the original polyhedron is adjacent to five other vertices, this means that each face of the dual polyhedron is adjacent to five other faces; in other words, each face of the dual polyhedron is a pentagon. We can also say that 80 of the dual polyhedron's vertices belong to 3 pentagons, while the other 12 vertices belong to 5 pentagons. (Hence the dual polyhedron is not semiregular. But it's still nice.)

HYPERBOLIC GEOMETRY

XI. For this question, "draw" means "draw, with reasonable accuracy, a picture using the Poincaré half-plane model of the hyperbolic plane". For this question, do not allow any of your hyperbolic lines to look like vertical Euclidean lines.

- (a) Draw three lines L, M, and N such that L and M are parallel to each other, L and N are ultraparallel to each other, and M and N intersect each other. (Be sure to label the lines correctly.)
- (b) Draw a quadrilateral ABCD such that each of its four angles is less than 30°. (Be sure that all four vertices really are points in the hyperbolic plane.)
- (c) Draw a line L through a point P, and draw the image of L under the isometry (of the hyperbolic plane) that is a rotation around P through a 120° angle. Mark the 120° angle in your drawing, and label the line L to distinguish it from its image line.



XII. Suppose that in the Poincaré half-plane, the line AB is an arc of a semicircle with one endpoint at the real number x. Prove that the horizontal translation $z \mapsto z - x$ followed by inversion in the unit circle is an isometry of the Poincaré half-plane that sends the line AB to a vertical line.

After the horizontal translation, the line AB is a Euclidean semicircle with one endpoint at the origin and the other at some real number w, say. The image of the whole circle under inversion in the unit circle is either a circle or a line. On the other hand, the image of the whole circle under inversion in the unit circle only intersects the real axis once, namely at the point 1/w. Furthermore, it is perpendicular to the x-axis there, since the original circle was perpendicular to the x-axis at w; in particular, the image cannot be a circle that it tangent to the x-axis. Since any circle that is not tangent to a line must intersect that line in 0 or 2 points, the image cannot be a circle at all; that is, the image must be a straight line. And since the original circle was perpendicular to the x-axis, the resulting line must also be perpendicular to the x-axis, that is, it is a vertical line. (One can also solve this problem simply by noting that a semicircle with one endpoint at the origin has the equation $(x-r)^2 + y^2 = r^2$ for some real number r, and computing the image of any point under inversion in the unit circle and noticing that the real part of the answer is the constant 1/2r.)

XIII. Find the distance between 15i and -7 + 8i in the Poincaré half-plane.

First we find the Euclidean semicircle that makes up the hyperbolic line between the two points 15i and -7 + 8i; that circle turns out to be centered at 8 and have radius 17, so that its endpoints are -9 and 25. Then we can translate horizontally by 9, sending the two points to 9 + 15i and 2 + 8i, and then invert in the unit circle, sending them to $\frac{1}{34} + \frac{5}{102}i$ and $\frac{1}{34} + \frac{2}{17}i$, respectively. The distance between the two points is thus $|\ln(\frac{2}{17}/\frac{5}{102})| = \ln\frac{12}{5}$. Or, we can simply calculate directly that the cross ratio (15i, -7 + 8i; -9, 25) equals $\frac{12}{5}$, which gives the distance as $|\ln(15i, -7 + 8i; -9, 25)| = \ln\frac{12}{5}$ as well.

XIV. Using the definitions of the hyperbolic trigonometry functions, prove that $\cosh^2 x - \sinh^2 x = 1$.

Since $\cosh x = (e^x + e^{-x})/2$ and $\sinh x = (e^x - e^{-x})/2$, we have

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$
$$= \frac{(e^x)^2 + 2e^x e^{-x} + (e^{-x})^2}{4} - \frac{(e^x)^2 - 2e^x e^{-x} + (e^{-x})^2}{4}$$
$$= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} = \frac{2 + 2}{4} = 1.$$

XV. Let $\triangle ABC$ be a right triangle with right angle $\angle C$. Suppose that $|CB| = \ln 5$ and $|CA| = \ln 13$. Verify, using the Hyperbolic Pythagorean Theorem $\cosh c = \cosh a \cosh b$, that $|AB| = \ln(17 + 12\sqrt{2})$.

We calculate:

$$\cosh a = \cosh(\ln 5) = \frac{e^{\ln 5} + e^{-\ln 5}}{2} = \frac{5 + 1/5}{2} = \frac{26/5}{2} = \frac{13}{5}$$
$$\cosh b = \cosh(\ln 13) = \frac{e^{\ln 13} + e^{-\ln 13}}{2} = \frac{13 + 1/13}{2} = \frac{170/13}{2} = \frac{85}{13},$$

and therefore $\cosh a \cosh b = (13/5)(85/13) = 17$; thus we need to check that $\cosh c = 17$ as well. Now

$$\cosh c = \cosh(\ln(17 + 12\sqrt{2}))$$
$$= \frac{e^{\ln(17 + 12\sqrt{2})} + e^{-\ln(17 + 12\sqrt{2})}}{2} = \frac{(17 + 12\sqrt{2}) + 1/(17 + 12\sqrt{2})}{2}.$$

We calculate that

$$\frac{1}{17+12\sqrt{2}} = \frac{17-12\sqrt{2}}{(17+12\sqrt{2})(17-12\sqrt{2})} = \frac{17-12\sqrt{2}}{17^2-(12\sqrt{2})^2} = \frac{17-12\sqrt{2}}{289-144\cdot 2} = 17-12\sqrt{2}.$$

Therefore

$$\cosh c = \frac{(17+12\sqrt{2})+1/(17+12\sqrt{2})}{2} = \frac{(17+12\sqrt{2})+(17-12\sqrt{2})}{2} = \frac{34}{2} = 17,$$

as asserted.