For this assignment, you may use anything in Chapters 1–3 and 6, *including* the assertions stated in the exercises. (You may always use things mentioned in class, problems from previous homeworks, and earlier problems from the same homework.) You may also use the facts—lemmas, theorems, corollaries—proved in Chapter 5 and Sections 7.1–7.7 and 7.11–7.13, but *not* the exercises from those sections (other than the ones mentioned in class, etc.), unless you prove them yourself.

Chapter 5

I. Baragar, p. 110, #5.11

II. Calculate the dihedral angle of a regular tetrahedron (that is, the angle between two of its faces).

Chapter 7

III. Baragar, p. 149, #7.34

IV. Let $A = i$, $B = 1 + i$, and $C = -1 + i$. Suppose that f is a direct isometry of the Poincaré half-plane model of hyperbolic geometry such that $f(A) = A$ and $f(B)$ is on the vertical line going through A. If $f(B)$ is above A, is the real part of $f(C)$ positive, negative, or zero? What if $f(B)$ is below A? (Hint: think about the angle $\angle BAC$.)

V. Write the fractional linear transformation $f(z) = -1/(4z + 10)$ as a composition using the functions $g(z) = -1/z$ and $h_\alpha(z) = z + \alpha$. You may use g and h multiple times in the composition, and you may use various values for α. (Hint: write everything in terms of matrices in $SL_2(\mathbb{R})$—but make sure the matrices you write down really *are* in $SL_2(\mathbb{R})$. The proof of Theorem 7.6.4 should be useful.)

VI. Suppose γ is a matrix in $SL_2(\mathbb{R})$ such that the fractional linear transformation T_γ fixes the point i. Show that

$$\gamma = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

for some real number θ. (Make sure you don't accidentally prove the converse. Hint: start with $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and figure out what relationships a, b, c, and d must satisfy.)

VII. What is the shortest path between the points $A = 12i$ and $B = 7 + 5i$ in the Poincaré half-plane? (Be specific.) Find an isometry that sends A and B to two points that lie on the same vertical line, and use this isometry to calculate the (hyperbolic) distance between A and B. By using the cross ratio with the points A and B directly, calculate the distance between them a second way.

VIII. Consider two points $A = x_1 + i$ and $B = x_2 + i$ in the Poincaré half-plane. Let ℓ_A and ℓ_B be the vertical lines going through A and B, respectively. Show that the angle between the (hyperbolic) lines ℓ_A and AB is

$$\arctan\left(\frac{2}{|x_1 - x_2|}\right),$$

and the same for the angle between the lines ℓ_B and AB. (Hint: consider the Euclidean picture these hyperbolic lines form.) Then show that the (hyperbolic) area of the (hyperbolic) quadrilateral whose vertices are

$$(-1 - \sqrt{2}) + i, \quad (1 - \sqrt{2}) + i, \quad (-1 + \sqrt{2}) + i, \quad \text{and } (1 + \sqrt{2}) + i$$

equals $\pi/2$. You may use the values $\tan(\pi/8) = \sqrt{2} - 1$ and $\tan(3\pi/8) = \sqrt{2} + 1$ without having to prove them. (Hint: draw vertical lines at each of the vertices, and consider all the quadrilateral's angles as sums or differences of the types of angles described earlier in the problem.)