Math 308, Section 101
Solutions for First Midterm
(October 6, 2004)

Note that there were two versions of the midterm, so some questions have two different variants.

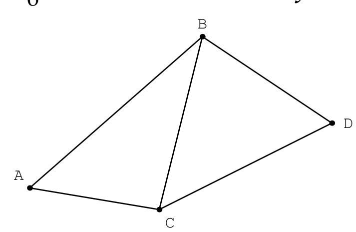
I. State the definition of "isometry".

An isometry is a function form the plane to itself that preserves distances. (This is already a correct answer, but we could further specify: f is an isometry if for any two points P and Q, the distance from P to Q is the same as the distance from $f(P)$ to $f(Q)$.)

II. State Axiom 5, sometimes called the Parallel Postulate. You may state either "Playfair's version" or "Euclid's version".

Playfair's version of Axiom 5 states: "Given a line ℓ and a point P not on ℓ, there exists a unique line ℓ_2 through P that is parallel to ℓ." Euclid's version of Axiom 5 states: "Suppose that a line ℓ meets two other lines ℓ_1 and ℓ_2 so that the sum of the interior angles on one side of ℓ is less than 180°. Then ℓ_1 and ℓ_2 intersect in a point on that side of ℓ."

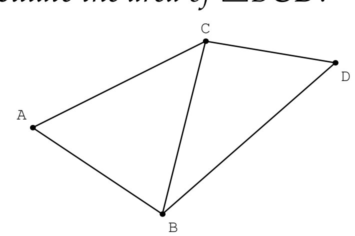
III. Triangles $\triangle ABC$ and $\triangle BCD$ share the side BC. If $|AC| = 6$, $|BD| = 7$, $|CD| = 9$, $\angle ABC = 30°$, and $\sin(\angle BAC) = \frac{5}{6}$, calculate the area of $\triangle BCD$.



By the Law of Sines, $\sin(\angle BAC)/|BC| = \sin(\angle ABC)/|AC|$, and therefore $|BC| = |AC|\sin(\angle BAC)/\sin(\angle ABC) = 6 \cdot \frac{5}{6}/\sin 30° = 5/\frac{1}{2} = 10$. This makes the semiperimeter of $\triangle BCD$ equal to $s = \frac{1}{2}(10+7+9) = 13$. By Heron's formula,

$$|\triangle ACD| = \sqrt{13(13-10)(13-7)(13-9)} = \sqrt{13 \cdot 3 \cdot 6 \cdot 4} = 3 \cdot 2 \cdot \sqrt{13 \cdot 2} = 6\sqrt{26}.$$

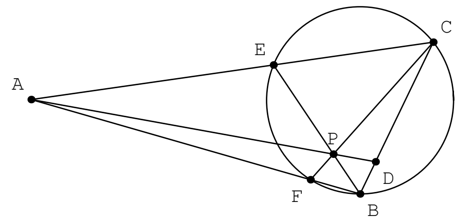
IV. Triangles $\triangle ABC$ and $\triangle BCD$ share the side BC. If $|AB| = 5$, $|AC| = 8$, $|BD| = 9$, $|CD| = 4$, and $\angle BAC = 60°$, calculate the area of $\triangle BCD$.



By the Law of Cosines, $|BC|^2 = |AB|^2 + |AC|^2 - 2|AB| \cdot |AC|\cos(\angle BAC) = 5^2 + 8^2 - 2 \cdot 5 \cdot 8 \cos 60° = 25 + 64 - 80 \cdot \frac{1}{2} = 49$, and hence $|BC| = 7$. This makes the semiperimeter of $\triangle BCD$ equal to $s = \frac{1}{2}(7+9+4) = 10$. By Heron's formula,

$$|\triangle ACD| = \sqrt{10(10-7)(10-9)(10-4)} = \sqrt{10 \cdot 3 \cdot 1 \cdot 6} = 2 \cdot 3 \cdot \sqrt{5} = 6\sqrt{5}.$$

V. Given $\triangle ABC$, let D, E, and F be points on the segments AB, AC, and BC, respectively. Suppose that B, C, E, and F all lie on a common circle, and that the cevians AD, BE, and CF intersect in a single point P. If $|AE| = 5$, $|AF| = 6$, $|CD| = 2$, and $|CE| = 3$, what is $|BD|$?



By the power of the point of A, we have $|AE| \cdot |AC| = |AF| \cdot |AB|$, and so $|AB| = |AE| \cdot |AC|/|AF| = 5(5+3)/6 = 20/3$, which means $|BF| = |AB| - |AF| = 20/3 - 6 = \frac{2}{3}$. Now by Ceva's Theorem,

$$\frac{|AF|}{|FB|}\frac{|BD|}{|DC|}\frac{|CE|}{|EA|} = 1,$$

which implies

$$|BD| = \frac{|FB| \cdot |DC| \cdot |EA|}{|AF| \cdot |CE|} = \frac{\frac{2}{3} \cdot 2 \cdot 5}{6 \cdot 3} = \frac{10}{27}.$$

VI. Given $\triangle ABC$, let D, E, and F be points on the segments AB, AC, and BC, respectively. Suppose that B, C, E, and F all lie on a common circle, and that the cevians AD, BE, and CF intersect in a single point P. If $|AE| = 8$, $|AF| = 10$, $|BD| = 1$, and $|BF| = 2$, what is $|CD|$?

(See diagram above.) By the power of the point of A, we have $|AE| \cdot |AC| = |AF| \cdot |AB|$, and so $|AC| = |AF| \cdot |AB|/|AE| = 10(10+2)/8 = 15$, which means $|CE| = |AC| - |AE| = 15 - 8 = 7$. Now by Ceva's Theorem,

$$\frac{|AF|}{|FB|}\frac{|BD|}{|DC|}\frac{|CE|}{|EA|} = 1,$$

which implies

$$|CD| = \frac{|AF| \cdot |BD| \cdot |CE|}{|FB| \cdot |EA|} = \frac{10 \cdot 1 \cdot 7}{2 \cdot 8} = \frac{35}{8}.$$

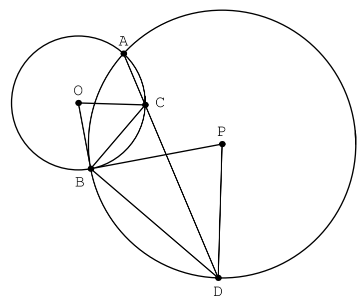
VII. Using the definition of $\cos A$, *prove that* $\cos 45^\circ = \sqrt{2}/2$. *Do not assume properties of any particular triangle unless you prove them.*

Let $\triangle ABC$ be a right triangle with right angle $\angle C$ and with both legs $|AC|$ and $|BC|$ having length 1. (If you don't trust that such a triangle exists, let two lines intersect perpendicularly, and draw a circle of radius 1 centered at the point of intersection; the right triangle just described can be made from various points of intersection.) Since $|AC| = |BC|$, we know that $\angle A = \angle B$ by *pons asinorum*. Now the sum of the three angles $\angle A + \angle B + \angle C$ equals 180°, or $\angle A + \angle B = 180^\circ - \angle C = 180^\circ - 90^\circ = 90^\circ$. Since $\angle A = \angle B$, this gives $2\angle A = 90^\circ$, or $\angle A = 45^\circ = \angle B$. By the Pythagoream Theorem, $|AB|^2 = |AC|^2 + |BC|^2 = 1^2 + 1^2 = 2$, which gives $|AB| = \sqrt{2}$. Therefore, by the definition of $\cos A$,

$$\cos A = \frac{|AC|}{|AB|} = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

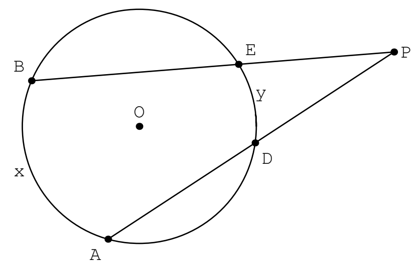
(Note: some were asked to prove that $\sin 45^\circ = \sqrt{2}/2$ instead, but the proof is essentially identical. Also note that you could scale $\triangle ABC$ differenly of course, and you could start by declaring various properties of $\triangle ABC$ and proving the others in a different order: for example, you could assume $\angle C = 90^\circ$, $\angle A = 45^\circ$, and $|AB| = 2$ and calculate the other two sides; or you could assume $|AC| = |BC| = 1$ and $|AB| = \sqrt{2}$ and calculate the angles.)

VIII. Let two circles centered at O and P intersect at the points A and B, as in the diagram. Let a line through A intersect the first circle at C and the second circle at D. Prove that $\triangle OBC \sim \triangle PBD$.



By the Star Trek Lemma, $\angle O = \text{arc } BC = 2\angle BAC$. Similarly, $\angle P = \text{arc } BD = 2\angle BAD$. Since $\angle BAC$ and $\angle BAD$ are the same angle, we see that $\angle O = \angle P$. Now $|OB| = |OC|$ since both are radii of the same circle, and similarly $|PB| = |PD|$. Therefore $|OB|/|PB| = |OC|/|PD|$, and hence $\triangle BOC \sim \triangle BPD$ by SAS for Similarity. (You could also use *pons asinorum* and the sum of the angles of a triangle equaling 180° to conclude that $\angle OBC = \angle OCB = \angle PBD = \angle PDB$.)

IX. Suppose an angle $\angle P$ is defined by two rays, the first of which intersects a circle $\mathcal{C}$ at the points A and D and the second of which intersects $\mathcal{C}$ at the points B and E, as shown in the diagram below. Let x be the angular measure of the arc AB, and let y be the angular measure of the arc DE. (In other words, $\angle AOB = x$ and $\angle DOE = y$.) Show that $\angle P = \frac{1}{2}(x - y)$.



Draw segment AE (or BD, either one works fine). By the Star Trek Lemma, $\angle AEB = \frac{1}{2}x$ and $\angle EAD = \frac{1}{2}y$. Since $\angle AEB$ is an exterior angle to $\triangle AEP$, we have $\angle AEB = \angle EAD + \angle P$, which gives $\angle P = \angle AEB - \angle EAD = \frac{1}{2}x - \frac{1}{2}y = \frac{1}{2}(x - y)$. (More longwinded solutions are also possible.)

X. Suppose that the centroid G of $\triangle ABC$ is the same as the circumcenter O of $\triangle ABC$. Prove that $\triangle ABC$ is equilateral.

Let D be the midpoint of the segment BC. Since the centroid G is the intersection of the three medians of $\triangle ABC$, we know that G is on the line AD. Also, since the circumcenter O is the intersection of the three perpendicular bisectors of the sides of $\triangle ABC$, we know that the line $OD = GD$ (since $O = G$) is perpendicular to BC. But the lines AD and GD share the two points G and D, and hence they must be the same line. In other words, AD is perpendicular to BC, and hence $\angle ADB = 90° = \angle ADC$. We also have $|BD| = |BC|$, since D is the midpoint of BC, and $|AD| = |AD|$ trivially. Therefore by SAS, $\triangle ADB = \triangle ADC$. We conclude that $|AB| = |AC|$. A similar argument using the midpoint of another side of $\triangle ABC$ proves that $|AB| = |BC|$. Therefore all three sides of $\triangle ABC$ have equal lengths, and so $\triangle ABC$ is equilateral. (One technical point, which I did not demand you address in your solutions, is that G and D might be the same point. But the centroid G is always $\frac{2}{3}$ of the way from a vertex to the opposite side, and so there is no way for the centroid to lie on the side of its triangle.)

XI. Suppose that the incenter I of $\triangle ABC$ is the same as the orthocenter H of $\triangle ABC$. Prove that $\triangle ABC$ is equilateral.

Let D be the point of intersection of the angle bisector of $\angle A$ and the segment BC. Since the incenter I is the intersection of the three angle bisectors of $\triangle ABC$, we know that I lies on the line AD. Let D' be the point of intersection of the altitude of $\triangle ABC$ at A with the line BC. Since the orthocenter H is the intersection of the three altitudes of $\triangle ABC$, we know that H lies on the line AD'. Now since $H = I$, the lines AD and AD' share the two points A and H, and therefore they must be the same line. Since this line can only intersect the line BC in a single point, we conclude that $D = D'$. Now $\angle BAD = \angle CAD$ since AD bisects $\angle A$, and $\angle ADB = 90° = \angle ADC$ since $AD = AD'$ is perpendicular to BC; also $|AD| = |AD|$ trivially. Therefore by ASA, $\triangle ADB = \triangle ADC$. We conclude that $|AB| = |AC|$. A similar argument using another vertex of $\triangle ABC$ proves that $|AB| = |BC|$. Therefore all three sides of $\triangle ABC$ have equal lengths, and so $\triangle ABC$ is equilateral. (One technical point, which I did not demand you address in your solutions, is that I and A might be the same point. But the incenter I always lies at the same perpendicular distance from each side of its triangle, and so there is no way for the incenter to lie on a side of its triangle.)