Math 308, Section 101
Solutions to Study Questions for First Midterm
(in class Wednesday, October 6, 2004)

I. State the three axioms concerning isometries (Axioms 6–8).

Axiom 6: For any two points P and Q, there exists an isometry f with $f(P) = Q$.
Axiom 7: For any three points P, Q, and R satisfying $|PQ| = |PR|$, there exists an isometry f with $f(P) = P$ and $f(Q) = R$.
Axiom 8: For any line ℓ, there is an isometry that fixes every point on ℓ and does not fix any other points.

II. State the definition of angle congruence.

$\angle ABC$ is congruent to $\angle A'B'C'$ if there exists an isometry f satisfying:

- $f(B) = B'$;
- $f(A)$ is on the ray $B'A'$;
- $f(C)$ is on the ray $B'C'$.

III. Using the definition of $\sin A$*, prove that* $\sin 60° = \sqrt{3}/2$*. Do not assume properties of any particular triangle unless you prove them.*

Draw an equilateral triangle $\triangle ABC$ with $|AB| = |BC| = |CA| = 1$. Let D be the midpoint of the segment of BC, so that AD is a median of $\triangle ABC$ and $|BD| = |DC| = \frac{1}{2}$. We also have $|AB| = 1 = |AC|$ and trivially $|AD| = |AD$. Therefore, by the SSS theorem, $\triangle ABD \equiv \triangle ACD$. This implies that $\angle ADB = \angle ADC$, and since those two angles are adjacent, we conclude that they are right angles, so that $\angle ADB = 90°$ and $\triangle ADB$ is a right triangle.

We also know that $\angle ABD = 60°$. (To see this, note that $\triangle ABC \equiv \triangle BCA$ by the SSS theorem. Therefore $\angle ABC = \angle BCA$ and also $\angle BCA = \angle CAB$. Since $\angle ABC + \angle BCA + \angle CAB = 180°$, we conclude that each angle equals $60°$.) Therefore $\sin 60° = |AD|/|AB|$ by the definition of sine. Now $|AB| = 1$ and $|BD| = \frac{1}{2}$, and so

$$|AD| = \sqrt{|AB|^2 - |BD|^2} = \sqrt{1^2 - \left(\tfrac{1}{2}\right)^2} = \sqrt{\tfrac{3}{4}} = \frac{\sqrt{3}}{2}$$

by the Pythagorean theorem. We conclude that $\sin 60° = \frac{\sqrt{3}}{2}/1 = \frac{\sqrt{3}}{2}$.

IV. (Baragar, p. 29, #1.45) In the quadrilateral $ABCD$ in Figure 1.20(b), AD is parallel to BC, $\angle C = 2\angle A$, $|CD| = 3$, and $|BC| = 2$. What is $|AD|$?

Extend the lines AB and CD to meet at P. Then $\angle BCD$ is an exterior angle of $\triangle BCP$, and so $\angle BCD = \angle CPB + \angle CBP$. On the other hand, since AD and BC are parallel, the corresponding angles $\angle A$ and $\angle CBP$ are equal, and hence $\angle BCD = 2\angle CBP$ by hypothesis. We therefore have

$$\angle CPB = \angle BCD - \angle CBP = 2\angle CBP - \angle CBP = \angle CBP.$$

We conclude from the converse to *pons asinorum* that $|CP| = |CB| = 2$, and so $|PD| = |PC| + |CD| = 2 + 3 = 5$.

Now we have $\angle PBC = \angle PAD$ and $\angle PCB = \angle PDA$ (corresponding angles of parallel lines) and $\angle P = \angle P$. Therefore $\triangle PBC \sim \triangle PAD$ by definition. By Corollary 1.7.4, we conclude that

$$\frac{|AD|}{|BC|} = \frac{|PD|}{|PC|}.$$

Since $|BC| = 2$, $|PD| = 5$, and $|PC| = 2$, we conclude that $|AD| = 5$.

V. (Baragar, p. 40, #1.76) In Figure 1.34(b), suppose $|AC'| = |C'B| = |CE| = 2$, $|CD| = 3$, and $|BF| = 1$. What is the area of $\triangle ABC$?

By the Power of the Point theorem (in fact, the version proved in problem IV of Homework #3) applied to B, we know that $|BC'||BA| = |BF||BD|$. We have $|BA| = |BC'| + |C'A| = 2 + 2 = 4$, and so $2 \cdot 4 = 1 \cdot |BD|$, yielding $|BD| = 8$. Since $|BD| = |BF| + |FC| + |CD| = 1 + |FC| + 3$, we have $|FC| = 4$. Now applying the Power of the Point theorem to C, we have $|AC||CE| = |FC||CD|$. We know three of these lengths, and so $|AC| \cdot 2 = 4 \cdot 3$, yielding $|AC| = 6$. We find that $\triangle ABC$ has side lengths $|AB| = 4$, $|AC| = 6$, and $|BC| = |BF| + |FC| = 1 + 4 = 5$. Therefore the semiperimeter is $s = \frac{1}{2}(6 + 4 + 5) = \frac{15}{2}$. By Heron's formula, the area of the triangle is

$$\begin{aligned} |\triangle ABC| = \sqrt{s(s-a)(s-b)(s-c)} &= \sqrt{\tfrac{15}{2}(\tfrac{15}{2} - 6)(\tfrac{15}{2} - 4)(\tfrac{15}{2} - 5)} \\ &= \sqrt{\tfrac{15}{2} \cdot \tfrac{3}{2} \cdot \tfrac{7}{2} \cdot \tfrac{5}{2}} = \frac{15\sqrt{7}}{4}. \end{aligned}$$

VI. (Baragar, p. 57, #1.123) Use Ceva's theorem to show that the altitudes intersect in a common point. (Hint: If CF is an altitude of $\triangle ABC$, then $|AF| = b\cos A \ldots$.)

We begin by assuming that $\triangle ABC$ has only acute angles, remarking on the other cases at the end. Let AD, BE, and CF be the altitudes of $\triangle ABC$; since the triangle is acute, all three points D, E, and F lie on the opposite side of the triangle (as opposed to on the extensions of those sides). Moreover, we have the following formulas, all straight from the definition of cosine:

$$\begin{aligned} |AF| &= b\cos A, & |FB| &= a\cos B, & |BD| &= c\cos B, \\ |DC| &= b\cos C, & |CE| &= a\cos C, & |EA| &= c\cos A. \end{aligned}$$

Therefore

$$\frac{|AF|}{|FB|}\frac{|BD|}{|DC|}\frac{|CE|}{|EA|} = \frac{b\cos A}{a\cos B}\frac{c\cos B}{b\cos C}\frac{a\cos C}{c\cos A} = 1.$$

Ceva's theorem now implies that the altitudes AD, BE, and CF all meet in a single point.

If, on the other hand, $\triangle ABC$ has an obtuse angle, say $\angle A > 90^\circ$ (a triangle can have at most one obtuse angle since the sum of all three angles is 180°), then the two altitudes BE and CF both meet the extended sides of the triangle rather than the sides themselves, while the altitude AD meets the side BC. Therefore exactly one of of the three points D, E, and F are on the sides of the triangle, and Ceva's theorem still applies. The only change to the above argument needed is to replace $\cos A$ with $|\cos A|$ everywhere.

(If $\triangle ABC$ has a right angle, say $\angle A$, then it is easy to see that all three altitudes intersect at A.)

VII. Let the quadrilateral $ABCD$ be a trapezoid: the sides AB and CD are parallel, but the sides AD and BC are not parallel. Suppose that $|AD| = |BC|$. Prove that $\angle ADC = \angle BCD$.

Since the lines AD and BC are not parallel, they meet in a single point P when extended. By Theorem 1.7.1, the fact that AB and CD are parallel implies that

$$\frac{|PA|}{|PD|} = \frac{|PB|}{|PC|}.$$

We first assume that C is between P and B and that D is between P and A, remarking on the other possible cases at the end. Since C is between P and B, we have $|PB| = |PC| + |CB|$; similarly, $|PA| = |PD| + |DA|$. Thus our previous equation becomes

$$1 + \frac{|DA|}{|PD|} = \frac{|PD| + |DA|}{|PD|} = \frac{|PC| + |CB|}{|PC|} = 1 + \frac{|CB|}{|PC|}.$$

Subtracting one from both sides and cross-multiplying, we get $|DA||PC| = |CB||PD|$. But $|AD| = |BC|$ by assumption, and so $|PC| = |PD|$. We conclude from *pons asinorum* that $\angle PCD = \angle PDC$, which implies that $\angle BCD = 180^\circ - \angle PCD = 180^\circ - \angle PDC = \angle ADC$ as desired.

If B is between P and C, then we instead get $|PC| = |PB| + |BC|$, or equivalently $|PB| = |PC| - |CB|$; similarly if A is between P and D, we have $|PA| = |PD| - |DA|$. The relevant equation is thus

$$1 - \frac{|DA|}{|PD|} = \frac{|PD| - |DA|}{|PD|} = \frac{|PC| - |CB|}{|PC|} = 1 - \frac{|CB|}{|PC|},$$

from which we again conclude that $|PC| = |PD|$ and hence $\angle PCD = \angle PDC$. In this case we are done immediately, since $\angle PCD$ and $\angle BCD$ are different names for the same angle, as are $\angle PDC$ and $\angle ADC$.

[The case where C is between P and B but A is between P and D is impossible, since we would get $|DA||PC| = -|CB||PD|$, which is impossible since all lengths involved are positive. The case where B is between P and C but D is between P and A is impossible for the same reason. Finally, P can never be between B and C or between A and D, for this would have the opposite sides BC and AD intersecting in the middle of a side.]

VIII. Let A, B, C, D, P, and Q be distinct points in the plane. Suppose that f is an isometry such that $f(A) = C$, $f(B) = D$, and $f(P) = Q$. Prove that P is on the perpendicular bisector of the segment AB if and only if Q is on the perpendicular bisector of the segment CD.

By problem IV on Homework #2, we know that the perpendicular bisector of a line segment is the same as the set of all points that are equidistant from the segment's endpoints. Therefore:

- P is on the perpendicular bisector of the segment AB if and only if $|AP| = |BP|$;
- Q is on the perpendicular bisector of the segment CD if and only if $|CQ| = |DQ|$.

However, the isometry f preserves distances; in particular, $|AP| = |f(A)f(P)| = |CQ|$ and $|BP| = |f(B)f(P)| = |DQ|$. Therefore:

- $|AP| = |BP|$ if and only if $|CQ| = |DQ|$.

The problem follows immediately from these three equivalences.

[One can certainly solve this problem by splitting it up into its "if" and "only if" directions and proving them separately.]

IX.

(a) Suppose that the incenter I of $\triangle ABC$ is the same as the centroid G of $\triangle ABC$. Prove that $\triangle ABC$ is equilateral.

(b) Suppose that the circumcenter O of $\triangle ABC$ is the same as the orthocenter H of $\triangle ABC$. Prove that $\triangle ABC$ is equilateral.

(a) Let D be the midpoint of BC, so that $|BD| = |DC|$. The centroid G is on the median AD by definition. Since $I = G$, we see that AD is also the angle bisector of $\angle BAC$. By the Angle Bisector Theorem (Exercise 1.43),

$$\frac{|AB|}{|AC|} = \frac{|BD|}{|DC|} = 1,$$

or $|AB| = |AC|$ in other words. The same argument, using the midpoint E of AC (so that BE is both a median and an angle bisector), shows that $|AB| = |BC|$ as well. Therefore, $\triangle ABC$ is equilateral.

(b) Let D be the midpoint of BC, so that $|BD| = |DC|$. Since the circumcenter O is on all of the perpendicular bisectors of $\triangle ABC$, and since there is only one line through D that is perpendicular to BC (this follows from Theorem 1.4.3), we know that OD is perpendicular to BC. On the other hand, we also know that $H = O$, where H is the orthocenter, the intersection of the altitudes. In particular, the line AH is the altitude perpendicular to BC.

Now both AH and OD are lines perpendicular to BC, and they meet at the common point $H = O$, and so again we conclude that the lines AH and OD are the same. In particular, AD is the perpendicular bisector of BC. We thus have $\angle ADB = \angle ADC = 90^\circ$, and also $|BD| = |DC|$ and $|AD| = |AD|$. Therefore, by the SAS theorem, $\triangle ADB \equiv \triangle ADC$. We conclude from this that $|AB| = |AC|$. The same argument, concentrating on the midpoint E of AC, shows that $|AB| = |BC|$ as well. Therefore, $\triangle ABC$ is equilateral.