Math 308, Section 101
Solutions for Homework #1
(due September 15, 2004)

I. Prove that the composition of two isometries is an isometry. In other words, suppose that the two functions f and g from the plane to itself are both isometries, and define another function h from the plane to itself by $h(P) = f(g(P))$; prove that h is an isometry.

To show that h is an isometry, we need to show that the definition of isometry holds for h, namely that for any two points P and Q, we have $d(h(P), h(Q)) = d(P, Q)$. Notice that $h(P) = f(g(P))$ and $h(Q) = f(g(Q))$; thus we need to prove that $d(f(g(P)), f(g(Q))) = d(P, Q)$.

Since g is an isometry, we know that

$$d(g(P), g(Q)) = d(P, Q).$$

Since f is an isometry, we know that $d(f(A), f(B)) = d(A, B)$ for any points A and B. Choosing $A = g(P)$ and $B = g(Q)$, we obtain

$$d(f(g(P)), f(g(Q))) = d(g(P), g(Q)).$$

These two equalities together imply that $d(f(g(P)), f(g(Q))) = d(P, Q)$, as desired.

II. Suppose that the isometry f fixes two distinct points A and B. Prove that it fixes every point C on the line AB. (Note: to say that f fixes a point P is simply to say that $f(P) = P$.) (Hint: use the statement of Exercise 1.11, which we mentioned in class. Split into cases depending on which of the three points A, B, C is between the other two.)

Let f be an isometry that fixes two distinct points A and B, and let C be any point on the line AB. We need to show that $f(C) = C$. If $C = A$ or $C = B$, then f fixes C by assumption; so we can assume that one of the following holds:

1. C is on the segment AB; or
2. A is on the segment BC; or
3. B is on the segment CA.

We show the proof of case 1; the other cases are very similar.

So suppose that C is on the segment AB. By Exercise 1.11, we know that $|AB| = |AC| + |CB|$. Define $C' = f(C)$; we want to show that $C' = C$. We know that $f(A) = A$ and $f(B) = B$; since f is an isometry, we have $d(A, C) = d(f(A), f(C)) = d(A, C')$ (in other words, $|AC| = |AC'|$) and similarly $|CB| = |C'B|$. By substituting into the earlier equation, it follows that $|AB| = |AC'| + |C'B|$.

By Exercise 1.11 again, we conclude that C' is on the segment AB. In particular, both C and C' are on the ray AB, and $|AC| = |AC'|$ as we have already seen. Since there is exactly one point on a ray at a given distance from its vertex (as said in class), the only possibility is that $C = C'$, as needed. (It is also possible to conclude that $C = C'$ from the version of Lemma 1.3.2 stated in class, since both C and C' lie on the line AB and also on the intersection of the circles $\mathcal{C}_{|AC|}(A)$ and $\mathcal{C}_{|CB|}(B)$.)

(In cases 2 and 3, the relevant starting equations are $|AB| = |BC| - |CA|$ and $|AB| = |CA| - |BC|$, respectively, both of which follow from Exercise 1.11 and a simple algebraic rearrangement. Otherwise, the proofs are essentially identical.)

III. Given a line ℓ and a point C that is not on ℓ, let f be the isometry that fixes every point on ℓ and no other points, and let $C' = f(C)$. Prove that ℓ is the perpendicular bisector of the segment CC'. Conclude that there exists a point D on ℓ such that the segment CD is perpendicular to ℓ.

Let f be the reflection through ℓ described in the statement of the problem (as guaranteed by Axiom 8), let C be a point not on ℓ, and let $C' = f(C)$. We know that C and C' are on opposite sides of ℓ, so let D be the intersection of the segment CC' and the line ℓ. We need to show that:

1. ℓ bisects CC', that is, $|CD| = |DC'|$; and
2. ℓ is perpendicular to CC'.

To prove #1, we note that $f(D) = D$ since f fixes all the points of ℓ. By the definition of isometry, we have $d(C, D) = d(f(C), f(D)) = d(C', D)$, which proves $|CD| = |C'D|$ as desired.

As for #2, let P be any point of ℓ other than D. Since f fixes the points of ℓ, we have $f(D) = D$ and $f(P) = P$. We also know that $f(C) = C'$. By the definition of angle congruence, we see that $\angle PDC = \angle PDC'$ (since certainly P is on the ray DP and C' is on the ray DC'). Since these two congruent angles are adjacent angles, we conclude from the definition of perpendicularity that ℓ and the line CC' are perpendicular. We also see now that D is a point on ℓ such that ℓ is perpendicular to CD, establishing the last assertion of the problem.

(Note that this problem is very similar to Exercise 1.17 in the textbook.)

IV. Prove the "SAS Theorem": if $\triangle ABC$ and $\triangle A'B'C'$ are triangles with $|AB| = |A'B'|$ and $|BC| = |B'C'|$ and with angle ABC congruent to angle $A'B'C'$, then $\triangle ABC \equiv \triangle A'B'C'$. (Hint: the fact that two angles are congruent means that there exists an isometry with certain properties—start there.)

Since $\angle ABC = \angle A'B'C'$, by the definition of angle congruence there exists an isometry f such that $f(B) = B'$, and $f(A)$ is on the ray $B'A'$ and also $f(C)$ is on the ray $B'C'$. On the other hand, $d(B', f(A)) = d(f(B), f(A)) = d(B, A)$ since f is an isometry, while $d(B, A) = d(B', A')$ by hypothesis. These two equalities imply that $d(B', f(A)) = d(B', A')$. Since both $f(A)$ and A' are on the ray $B'A'$, and since there is only one point on a ray at a given distance from its vertex, we conclude that $f(A) = A'$.

The same argument (changing all the A's to C's) shows that $f(C) = C'$ as well. Since $f(A) = A'$, $f(B) = B'$, and $f(C) = C'$, the definition of triangle congruence tells us that $\triangle ABC \equiv \triangle A'B'C'$.