Math 308, Section 101
Solutions for Homework #2
(due September 22, 2004)

I. Prove that if $\triangle ABC \equiv \triangle A'B'C'$, then $\angle A = \angle A'$. (In other words, given that the definition of triangle congruence is satisfied, prove that the definition of angle congruence is satisfied.)

Since $\triangle ABC \equiv \triangle A'B'C'$, there exists an isometry f such that $f(A) = A'$, $f(B) = B'$, and $f(C) = C'$. In particular, $f(A) = A'$; and $f(B) = B'$ certainly lies on the ray $A'B'$, and $f(C) = C'$ certainly lies on the ray $A'C'$. Therefore the isometry f satisfies the three necessary properties in the definition of angle congruence, and so $\angle BAC = \angle B'A'C'$ as desired. (This problem wasn't assigned because it was difficult—it was assigned to make the point that we can't rely on "intuitive" facts about geometry unless we prove them.)

II. (Baragar, p. 24, #1.34) Suppose two lines intersect at P inside a circle and meet the circle at A and A' and at B and B', as shown in Figure 1.14(a). Let α and β be the measures of the arcs $A'B'$ and AB respectively. Prove that

$$\angle APB = \frac{\alpha + \beta}{2}.$$

Draw the segment AB'. By the Star Trek Lemma, $\angle AB'B$ equals half the angular measure of the arc AB, and so $\angle AB'B = \beta/2$; similarly, $\angle B'AA'$ equals half the angular measure of the arc $A'B'$, and so $\angle B'AA' = \alpha/2$. Now $\angle APB$ is the exterior angle to the triangle $\triangle AB'P$, and so its measure equals the sum of the measures of the other two interior angles (Corollary 1.4.7). Therefore,

$$\angle APB = \angle AB'P + \angle B'AP = \angle AB'B + \angle B'AA' = \frac{\beta}{2} + \frac{\alpha}{2} = \frac{\alpha + \beta}{2},$$

as claimed.

III. (Baragar, p. 25, #1.40) Let E be a point inside a square $ABCD$ such that $\triangle BCE$ is an equilateral triangle, as in Figure 1.16(b). Show that $\angle EAD = 15^\circ$.

We first remark that all three angles of an equilateral triangle $\triangle BCE$ must measure 60°. To see this, note that $|BC| = |BE|$ since the triangle is equilateral, and so $\angle BCE = \angle BEC$ by *pons asinorum*. Similarly, $|CB| = |CE|$ and so $\angle CBE = \angle CEB$ by *pons asinorum*. Therefore all three angles of $\triangle BCE$ are congruent, and their measures sum to 180° by Theorem 1.4.6; we conclude that each of the three angles is 60°. (Another way of showing this is by using the SSS theorem to prove that $\triangle BCE \equiv \triangle CEB$, for example.)

Now $\angle ABC$ is a right angle since $ABCD$ is a square, and so $\angle ABE = \angle ABC - \angle BCE = 90^\circ - 60^\circ = 30^\circ$. Since the measures of the angles of the triangle $\triangle ABE$ sum to 180°, we see that

$$\angle BEA + \angle BAE = 150^\circ.$$

On the other hand, $|BA| = |BC|$ since $ABCD$ is a square, while $|BC| = |BE|$ since $\triangle BCE$ is equilateral. Therefore $|BA| = |BE|$. By *pons asinorum* once more, we see that

$$\angle BEA = \angle BAE.$$

These two equations together imply that $\angle BAE = 75^\circ$. Since $\angle EAD$ is a right angle, again by the fact that $ABCD$ is a square, we conclude that $\angle EAD = \angle BAD - \angle BAE = 90^\circ - 75^\circ = 15^\circ$, as asserted.

IV. Show that the perpendicular bisector of a line segment is the set of all points equidistant from the two endpoints of the segment. In other words, let the line ℓ be the perpendicular bisector of the segment AB. For any point C, prove that C is on the line ℓ if and only if $|AC| = |BC|$.

Suppose first that C is a point on ℓ; we want to show that $|AC| = |BC|$. Let M be the intersection of the segment AB and the line ℓ. Since ℓ bisects the segment AB, we have $|AM| = |BM|$. Also, $\angle AMC = \angle BMC$ since both are right angles, and clearly $|CM| = |CM|$. Therefore, by the SAS Theorem, the two triangles $\triangle AMC$ and $\triangle BMC$ are congruent. In particular, the corresponding sides AC and BC are congruent, that is, $|AC| = |BC|$. (It is also possible to show this using the Pythagorean Theorem.)

Conversely, suppose that $|AC| = |BC|$; we want to show that C is on the line ℓ. Again, we have $|AM| = |BM|$ and $|CM| = |CM|$. Therefore, by the SSS Theorem, the two triangles $\triangle AMC$ and $\triangle BMC$ are congruent. In particular, the corresponding angles $\angle AMC$ and $\angle BMC$ are congruent, by Problem I. Since these two angles are adjacent, we conclude that both are right angles and that AB is perpendicular to MC. Now both ℓ and the line MC are perpendicular to the line AB. If ℓ and MC were not the same line, then they would be parallel by Theorem 1.4.3; however, they intersect at M, so they can't be parallel. The only possibility is that ℓ and MC are indeed the same line; in particular, C is a point on ℓ, as claimed.

V. Suppose that $\mathcal{C}$ is a circle with center O, that P is a point on the circle $\mathcal{C}$, and that ℓ is a line through P. Prove that ℓ is tangent to $\mathcal{C}$ if and only if ℓ is perpendicular to the radius OP of $\mathcal{C}$.

Suppose first that ℓ is perpendicular to OP. Let Q be any point on ℓ other than P. Then by the Pythagorean Theorem, $|OQ|^2 = |OP|^2 + |PQ|^2$. In particular, $|PQ|^2 > 0$, so $|OQ|^2 > |OP|^2$ and hence $|OQ| > |OP|$. Therefore the distances from O to Q and P are different, so Q cannot be on the circle centered at O that goes through P, which is $\mathcal{C}$ itself. We have shown that no other point on ℓ other than P lies on $\mathcal{C}$; therefore ℓ is tangent to $\mathcal{C}$ by definition.

Suppose now that ℓ is not perpendicular to OP. Choose a point D on ℓ such that OD is perpendicular to ℓ (as guaranteed by problem III on Homework #1). Let f be the reflection through the line OD, as described in Axiom 8. Since P is not on the line OD, the isometry f does not fix P but sends it to some other point P'. On the other hand, since ℓ is perpendicular to OD, points on ℓ are sent to other points on ℓ under this reflection, and so P' lies on ℓ. Moreover, since f is an isometry, $d(O,P) = d(f(O), f(P)) = d(O,P')$, and so P' lies on the circle $\mathcal{C}$. In summary, we have found a second point $P' \neq P$ on the intersection of ℓ and $\mathcal{C}$, and so ℓ is not tangent to $\mathcal{C}$.

(Notice that for the second half of the proof, we established "if ℓ is tangent to $\mathcal{C}$, then ℓ is perpendicular to OP" by proving the contrapositive, namely "if ℓ is not perpendicular to OP, then ℓ is not tangent to $\mathcal{C}$".)

VI. (Baragar, p. 23, #1.33) Suppose AT is a line segment that is tangent to a circle. Prove that $\angle ATB$ is half the measure of the arc TB which it subtends. Do this by picking a point C on the circle such that $\angle TCB$ subtends the arc TB (as in Figure 1.13(b)). Show that $\angle ATB = \angle TCB$. (Hint: don't just pick a random point C—pick an especially convenient point C.)

Note that it doesn't matter what point C we choose—the Star Trek Lemma tells us that the measure of $\angle TCB$ is half the angular measure of the arc TB, no matter what C is. So we choose C directly opposite B on the circle, so that BC is a diameter of the circle; in particular, the center O of the circle lies on the segment BC.

Again by the Star Trek Lemma, $\angle TBC$ equals half the angular measure of the arc TC, and $\angle TCB$ equals half the angular measure of the arc TB. But together, the angular measures of the arcs TC and TB add to $180°$, since BC is a diameter. Therefore, $\angle TBC + \angle TCB = 90°$, which we rewrite as $\angle TCB = 90° - \angle TBC$.

Now $OT = OB$ since both are radii of the same circle; therefore, by *pons asinorum*, we have $\angle TBC = \angle OTB$. Substituting this into the previous equation gives

$$\angle TCB = 90° - \angle OTB.$$

On the other hand, since TA is tangent to the circle at P, we know from Problem V that TA and OT are perpendicular. In particular, this gives $\angle ATB + \angle BTO = \angle ATO = 90°$, or

$$\angle ATB = 90° - \angle OTB.$$

Comparing these two equations, we conclude that $\angle ATB = \angle TCB$.

(Another similarly straightforward solution starts by choosing C so that TC, rather than BC, is a diameter. Yet another solution, which doesn't depend on C being chosen in a particular way, is to use *pons asinorum* on the triangle $\triangle OTB$, and the Star Trek Lemma to compare $\angle TCB$ and $\angle TOB$, to show that $\angle OTB = 90° - \angle TCB$; problem V tells us that $\angle OTB = 90° - \angle ATB$ as well.)