Math 308, Section 101
Solutions for Homework #3
(due September 29, 2004)

I. (Baragar, p. 28, #1.42) Let $\triangle ABC$ be an arbitrary triangle. Let A', B', and C' be the midpoints of the opposite sides. Draw lines through A', B', and C' that make an angle of 60° with each side, as in Figure 1.19(a). These lines intersect at A'', B'', and C'' as shown. Prove that $\triangle A''B''C''$ is similar to $\triangle ABC$.

We use the fact that the four angles in a quadrilateral add to 360°. In particular, $\angle A'BC' + \angle BC'B'' + \angle C'B''A' + \angle B''A'B = 360^\circ$. But $\angle B''A'B = 60^\circ$ by assumption, and $\angle BC'B'' = 180^\circ - \angle AC'A'' = 180^\circ - 60^\circ = 120^\circ$. Therefore $\angle A'BC' + \angle C'B''A' = 360^\circ - 60^\circ - 120^\circ = 180^\circ$. On the other hand, $\angle C'B''A' + \angle A'B''A'' = 180^\circ$ as well. We conclude that $\angle A'BC' = \angle A'B''A'$, which is to say that $\angle CBA = \angle C''B''A''$. The same reasoning proves that $\angle ACB = \angle A''C''B''$, and since the angles in both triangles $\triangle ABC$ and $\triangle A''B''C''$ must add to 180°, we see that $\angle BAC = \angle B''A''C''$ as well. This shows that $\triangle ABC \sim \triangle A''B''C''$, as desired.

(Note that we never used the fact that the three indicated angles measured exactly 60°—we only needed them all to be congruent to one another. We also never used at all the fact that A', B', and C' were the midpoints of their sides; any points on those sides would be fine.)

II. (Baragar, p. 29, #1.43) In an arbitrary triangle $\triangle ABC$, let the interior angle bisector at A intersect the side BC at D. Show that

$$\frac{|BD|}{|DC|} = \frac{|AB|}{|AC|}.$$

Hint: Construct the line parallel to AD and through B, as shown in Figure 1.19(b). let this intersect AC at E. Show $|AB| = |AE|$.

Following the hint, we consider the triangle $\triangle ABE$. Since EB and AD are parallel, we have $\angle ABE = \angle BAD$ (opposite interior angles) and $\angle AEB = \angle CAD$ (corresponding angles). But D was chosen so that $\angle BAD = \angle CAD$, and so we conclude that $\angle AEB = \angle ABE$. Therefore, by the converse to *pons asinorum* (Exercise 1.24), we have $|AE| = |AB|$.

On the other hand, by Theorem 1.7.1, we find that

$$\frac{|CE|}{|CA|} = \frac{|CB|}{|CD|}.$$

Because $|CE| = |CA| + |AE|$ and $|CB| = |CD| + |DB|$ by the "equality" case of the triangle inequality, this becomes

$$1 + \frac{|AE|}{|CA|} = \frac{|CA| + |AE|}{|CA|} = \frac{|CD| + |DB|}{|CD|} = 1 + \frac{|DB|}{|CD|}.$$

Finally, we may subtract 1 from each side of the equation and use the substitution $|AE| = |AB|$ from the first paragraph to get

$$\frac{|AB|}{|CA|} = \frac{|DB|}{|CD|},$$

which finishes the proof.

III. (Baragar, p. 29, #1.44) The pentagon in Figure 1.20(a) is regular and each side has length one. Show that

$$\frac{|AF|}{|FD|} = \frac{1+\sqrt{5}}{2}.$$

As a first step, we prove that the segments AC and ED are parallel. We saw in class that each angle of a regular pentagon measures 108°. In particular, $\angle BAC$ and $\angle BCA$ must add to $180^\circ - \angle ABC = 180^\circ - 108^\circ = 72^\circ$. However, since $|AB| = |BC|$, the triangle $\triangle ABC$ is isosceles, and so $\angle BAC = \angle BCA$, which forces $\angle BAC = \frac{1}{2} \cdot 72^\circ = 36^\circ$. Therefore $\angle CAE = \angle BAE - \angle BAC = 108^\circ - 36^\circ = 72^\circ$. Finally, we have $\angle CAE + \angle AED = 72^\circ + 108^\circ = 180^\circ$, which implies at last that AC and ED are parallel (this follows from Euclid's version of Axiom 5, as described in class).

Now that AC and ED are parallel, we quickly see that $\angle CAF = \angle FDE$ and $\angle ACF = \angle FED$ (alternate interior angles) amd that $\angle AFC = \angle DEF$ (vertical angles), this proving that $\triangle ACF \sim \triangle DEF$. Therefore by Corollary 1.7.4,

$$\frac{|AF|}{|FD|} = \frac{|AC|}{|ED|} = \frac{|AC|}{1} = |AC|.$$

On the other hand, because the pentagon is regular, we certainly have $|AE| = |AB|$, $\angle AED = \angle ABC$, and $|ED| = |BC|$, and therefore $\triangle AED \equiv \triangle ABC$ by the SAS theorem. It follows that $|AC| = |AD|$. But certainly $|AD| = |AF| + |FD|$ by the "equality" case of the triangle inequality. Our equation thus becomes

$$\frac{|AF|}{|FD|} = |AF| + |FD|.$$

Finally, the argument in the first paragraph extends to show that $\angle BAC = \angle BCA = \angle DAE = \angle ADE = \angle ECD = \angle CED = 36^\circ$. Then $\angle CAF = \angle EAB - \angle DAE - \angle CAB = 108^\circ - 36^\circ - 36^\circ = 36^\circ$, and the same argument applies for $\angle ACF$. Now we can say that $\angle CAF = 36^\circ = \angle CAB$, $\angle ACF = 36^\circ = \angle ACB$, and $|AC| = |AC|$, and so $\triangle ACF \equiv \triangle ACB$ by the ASA theorem. In particular, $|AF| = |AB| = 1$, and so our equation becomes

$$\frac{1}{|FD|} = 1 + |FD|,$$

or equivalently (multipling both sides by $|FD|$ and subtracting 1)

$$0 = |FD|^2 + |FD| - 1.$$

We have shown that the length $|FD|$ is a root of the quadratic equation $x^2 + x - 1 = 0$. The two roots of this equation are $(\sqrt{5}-1)/2$ and $(-\sqrt{5}-1)/2$. Since $|FD|$ is positive, it must be the case that $|FD| = (\sqrt{5}-1)/2$. Finally,

$$\frac{|AF|}{|FD|} = |AF| + |FD| = 1 + \frac{\sqrt{5}-1}{2} = \frac{\sqrt{5}+1}{2},$$

as claimed. (Note that $|AC| = 2\cos 36^\circ$, and so this exercise actually *proves* that $\cos 36^\circ = (1+\sqrt{5})/4$.)

IV. (Baragar, p. 31, #1.51) Theorem 1.8.1 is also true if the point P lies outside the circle, as in Figure 1.25(a). Prove it for this case.

Since $QRR'Q'$ is a cyclic quadrilateral (that is, all four vertices lie on a circle), its opposite angles $\angle QRR'$ and $\angle QQ'R'$ sum to 180° by Exercise 1.36. On the other hand, $\angle QRR'$ and $\angle QRP$ sum to 180° as well, and therefore $\angle QQ'R' = \angle QRP$. By the same argument, $\angle RR'Q' = \angle RQP$. Of course $\angle QPR = \angle QPR$ as well, and so $\triangle PQR \sim \triangle PR'Q'$ by the definition of similarity. By Corollary 1.7.4, we can conclude that

$$\frac{|PQ|}{|PR'|} = \frac{|PR|}{|PQ'|},$$

or $|PQ||PQ'| = |PR||PR'|$.

V. Let $ABCD$ be a quadrilateral. Prove that $ABCD$ is a parallelogram if and only if the segments AC and BD bisect each other.

First we assume that $ABCD$ is a parallelogram and prove that the diagonals AC and BD bisect each other. We begin by showing that the opposite sides AB and CD are congruent. Since AB and CD are parallel, the opposite interior angles $\angle ABD$ and $\angle CDB$ are congruent. Since AD and BC are parallel, we also have $\angle ADB = \angle CBD$. Clearly $|BD| = |DB|$ as well, and so by the ASA theorem, $\triangle ABD \equiv \triangle CDB$. We conclude that $|AB| = |CD|$.

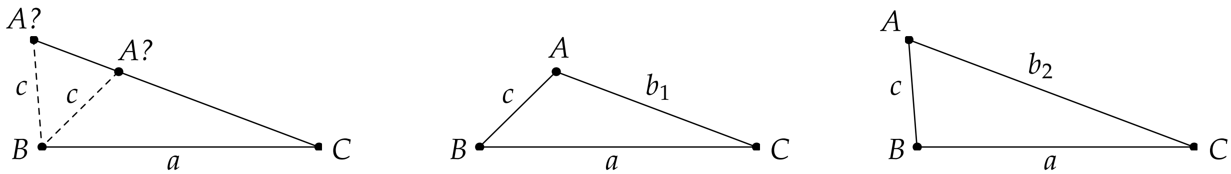
Now let P be the intersection of the diagonals AC and BD. Again we have two pairs of opposite interior angles that are thus congruent, namely $\angle ABP = \angle CDP$ and $\angle BAP = \angle DCP$. Furthermore, the vertical angles $\angle APB$ and $\angle CPD$ are congruent as well. Therefore, $\triangle ABP \sim \triangle CDP$. By Corollary 1.7.4, we conclude that

$$\frac{|AB|}{|CD|} = \frac{|AP|}{|CP|} = \frac{|BP|}{|DP|}.$$

However, we showed already that $|AB| = |CD|$. Therefore $|AB|/|CD| = 1$, which implies that $|AP| = |CP|$ and $|BP| = |DP|$. This shows that the segments AC and BD bisect each other.

Now we assume that AC and BD bisect each other and prove that $ABCD$ is a parallelogram. By hypothesis, we have $|AP| = |CP|$ and $|BP| = |DP|$. Also, the vertical angles $\angle APB$ and $\angle CPD$ are congruent. Therefore $\triangle ABP \equiv \triangle CDP$ by the SAS theorem. In particular, $\angle ABP = \angle CDP$, and so the lines AB and CD are parallel by Corollary 1.4.4. The same argument, applied to the triangles $\triangle ADP$ and $\triangle BCP$, shows that AD and BC are parallel.

VI. Recall that there is no "SSA theorem". In other words, let the lengths c and a of two sides of a triangle and the measure of $\angle C$ be given (for the purposes of this problem, we will assume that $a > c$); then the length of the third side b is generally not uniquely determined. However, there are only two possible values for b, corresponding to the two possible locations for the vertex A relative to B and C. (You don't have to prove this.) Call the two possible values b_1 and b_2.



(a) Prove that $b_1 b_2 = a^2 - c^2$ by using the power of the point C with respect to the circle $\mathcal{C}_c(B)$.
(b) Prove that $b_1 b_2 = a^2 - c^2$ by using the Law of Cosines.
(c) Find and prove a formula for $b_1 + b_2$ in terms of a and $\angle C$.

(a) Draw the circle of radius c around the point B. It intersects the ray coming out of C in the two points marked A? in the diagram, which are at distances b_1 and b_2 from C. Therefore, with respect to the circle $\mathcal{C}_r(B)$, we have $\Pi(C) = b_1 b_2$. On the other hand, the same circle intersects the line BC in the two points that are at distance c from B. The distances of these two points from C are thus $a - c$ and $a + c$. Therefore $\Pi(C) = (a - c)(a + c) = a^2 - c^2$ as well, proving that $b_1 b_2 = a^2 - c^2$.

(b) The Law of Cosines is $c^2 = a^2 + b^2 - 2ab \cos C$, which can be rewritten as $b^2 - (2a \cos C)b + (a^2 - c^2) = 0$. Consider the expression $f(b) = b^2 - (2a \cos C)b + (a^2 - c^2)$ as a polynomial in b. Both b_1 and b_2 make the equation $f(b) = 0$ valid (because they result in valid triangles); in other words, they are both roots of the polynomial $f(b)$. But the product of the roots of the polynomial $x^2 + Mx + N$ is simply N; in this case, the product $b_1 b_2$ of the roots must be $a^2 - c^2$. (Note: this fact about the product of the roots of a quadratic polynomial should be well-known from algebra, and it can also be recovered by expanding out $(x - r_1)(x - r_2)$ where r_1 and r_2 are the roots.)

(c) The sum of the roots of a polynomial $x^2 + Mx + N$ is $-M$. Therefore, by the same argument as in part (b), we can immediately deduce that $b_1 + b_2 = 2a \cos C$. (The same conclusion can be formed by drawing the line through B perpendicular to the ray out of C, which bisects the segment between the two points called A? in the diagram.)