Math 308, Section 101
Solutions for Homework #4
(due October 20, 2004)

I. (Baragar, p. 73, #3.5) Prove Lemma 3.3.5 (given $\triangle ABC$ and two points A' and D, it is possible to construct a triangle $\triangle A'B'C'$ that is congruent to $\triangle ABC$ such that B' is on the line $A'D$) and its corollary (we can reproduce a constructed angle on any constructed line).

It is possible to prove the corollary first and then prove the lemma from it, but we will do it in the order described. Start by constructing $\mathcal{C}_{A'}(|AB|)$, the circle centered at A' of length $|AB|$, which is possible by Lemma 3.3.3. This circle intersects the line $A'D$ in two points; call one of them B'. Now construct $\mathcal{C}_{A'}(|AC|)$ and $\mathcal{C}_{B'}(|BC|)$, again by Lemma 3.3.3. These circles intersect in two points; call one of them C'. By construction, we have $|A'B'| = |AB|$, $|A'C'| = |AC|$, and $|B'C'| = |BC|$, and therefore $\triangle A'B'C' \equiv \triangle ABC$ by the SSS theorem. This establishes Lemma 3.3.5. As for its corollary, let $\angle BAC$ be already constructed, and suppose we want to make a congruent angle on the line $A'D$. Simply make a congruent copy $\triangle A'B'C'$ of $\triangle ABC$ such that B' is on the line $A'D$ as per Lemma 3.3.5; then $\angle B'A'C' = \angle BAC$ as desired.

II. Given a constructed line AB and a constructed point C, describe how to construct a line through C that is perpendicular to AB. (Note: do not make any assumption about whether or not C is on the line AB.)

Without loss of generality, we can assume that $C \neq A$ (if $C = A$, then $C \neq B$ and we can just switch the names of A and B). Construct $\mathcal{C}_C(|CA|)$, the circle centered at C that goes through A. Now a circle intersects a line in 0, 1, or 2 points; the circle just constructed has at least one point A of intersection with the line AB. If it has only this one point of intersection, then the line AB is tangent to the circle by the definition of tangency. By a previous homework problem, this implies that the radius CA is perpendicular to the line AB, and we're already done with the desired construction.

So let's assume that the line AB and the circle $\mathcal{C}_C(|CA|)$ intersect in a second point A' as well. Construct the midpoint M of the segment AA'. (If C is on the line AB, then $C = M$ and the perpendicular bisector of AA' is the desired line. Hence we can assume from now on that C is not on the line AB.) Note that $|CA| = |CA'|$ since both A and A' lie on the same circle centered at C, and $|AM| = |A'M|$ since M is the midpoint of AA'. The side CM is shared, and so we conclude that $\triangle CAM \equiv \triangle CA'M$ by SSS. In particular, $\angle CMA = \angle CMA'$, and so CM is perpendicular to the original line AA' as desired.

III. (Baragar, p. 79, #3.17) Construct a segment of length $\sqrt{5}$, and from this, construct a segment of length $\sqrt{5} - 1$. Use it to construct a regular pentagon inscribed in a circle of radius four.

Begin by constructing any two perpendicular lines intersecting at A. On one line, construct a point B such that $|AB| = 1$ (that is, A and B are the same distance apart as the base points O and P), and on the other line, construct a point C such that $|AC| = 2$. (To do the latter, let $\mathcal{C}_A(|OP|)$ cut this second line at C', and then let $\mathcal{C}_{C'}(|OP|)$ cut the line at C, so that $|AC| = |AC'| + |C'C| = |OP| + |OP| = 1 + 1 = 2$.) Then by the Pythagorean Theorem, $|BC| = \sqrt{|AB|^2 + |AC|^2} = \sqrt{5}$. We can immediately construct the circle through A centered at B, which cuts the segment BC at the point D, say; then $|CD| = |BC| - |BD| = \sqrt{5} - 1$.

Now construct a segment of length 4. (Again, we can start with the segment AC of length 2, and construct $\mathcal{C}_A(|AC|)$ which cuts the line AC again at E; then $|CE| = |CA| + |AE| = 2 + 2 = 4$.) Use this to construct a circle $\mathcal{C}_E(|CE|)$ of radius four. Also construct the point H between E and C such that $|EH| = |CD| = \sqrt{5} - 1$. Finally, as in the end of the book's construction of the regular pentagon, construct the line perpendicular to EC at H, which intersects $\mathcal{C}_E(|CE|)$ in the points F and F', and construct the circles $\mathcal{C}_F(|BD|)$ and $\mathcal{C}_{F'}(|BD|)$ which intersect $\mathcal{C}_E(|CE|)$ again in the points G and G'. Then $CFGG'F'$ is a regular pentagon.

IV. Suppose that the orbits of Venus and Earth are perfectly circular with the sun at the center. Venus is usually very close to the sun (hence its being known as both the "evening star" and the "morning star"). Suppose that through years of observation, you decide that the farthest away from the sun that Venus ever looks is $45°$ apart in the sky. Calculate, with justification, the distance of Venus from the sun.

Let S, E, and V represent the positions of the sun, Earth, and Venus, respectively. As the two planets revolve around the sun, their orbits form circles centered at S. The angle between the sun and Venus referred to in the problem is simply $\angle SEV$ (think of the observer on Earth as the vertex, arms pointing up at the sun and Venus to form the sides of the angle). We are given that the most this angle ever measures is $45°$. This occurs when the line EV is tangent to the orbit of Venus. (Because the problem is stated not in complete formality, this observation can be advanced without formal justification. Ultimately a formal proof would rely on very technical facts about betweenness of points.) By a previous homework problem, the line EV is tangent to the orbit, which is simply $\mathcal{C}_S(|SV|)$, precisely when EV is perpendicular to the radius SV.

We now have a triangle $\triangle ESV$ where $\angle E = 45°$ and $\angle V = 90°$. By definition, $\sin E = |SV|/|ES|$, or $|SV| = |ES| \sin 45° = |ES|\sqrt{2}/2$. Using any reasonable source for the distance from the Earth to the sun allows us to finish the computation. For example, the table on page 64 of the textbook gives $|ES| = R_S = 149$ million kilometres. From this we compute that $|SV| = 149\sqrt{2}/2 \approx 105$ million kilometres. (In reality, this is a bit smaller than the actual minimum distance from Venus to the sun, which is about 107.48 million kilometres. The maximum distance between Venus and the sun is about 108.94 million kilometres.)

V. Suppose you have already constructed a circle $\mathcal{C}$ centered at O and going through a point P, and two line segments AB and DE. Describe, with justification, a construction of a circle $\mathcal{C}'$ such that

$$\frac{|\mathcal{C}|}{|\mathcal{C}'|} = \frac{|AB|}{|DE|},$$

where $|\mathcal{C}|$ and $|\mathcal{C}'|$ denote the areas enclosed by the circles $\mathcal{C}$ and $\mathcal{C}'$, respectively.

If we can construct a segment YZ of length $|OP|\sqrt{|DE|/|AB|}$, then the circle $\mathcal{C}'$ centered at Y going through Z will have area

$$|\mathcal{C}'| = \pi|YZ|^2 = \pi\left(|OP|\sqrt{\frac{|DE|}{|AB|}}\right)^2 = \pi|OP|^2\frac{|DE|}{|AB|} = |\mathcal{C}|\frac{|DE|}{|AB|},$$

which will give $|\mathcal{C}|/|\mathcal{C}'| = |AB|/|DE|$ as desired. So: using Lemma 3.4.3, we can start with a segment of length $|AB|$ and construct a segment of length $1/|AB|$. Then using Lemma 3.4.2, we can construct a segment of length $|DE| \cdot (1/|AB|) = |DE|/|AB|$, and by Lemma 3.4.5 we can construct a segment of length $\sqrt{|DE|/|AB|}$. Finally, by Lemma 3.4.2 again we can construct a segment of length $|OP|\sqrt{|DE|/|AB|}$, which yields the circle we want.

VI. Suppose you have already constructed a square $ABCD$ and a segment EF with $|EF| > |AB|$. Describe, with justification, a construction of a rhombus $WXYZ$ with side length $|WX| = |EF|$ whose area is the same as the area of $ABCD$.

It seems to be most straightforward to try to determine the lengths of the diagonals of the desired rhombus. First we claim that the diagonals of any rhombus $WXYZ$ are perpendicular to each other and bisect each other. To see this, consider first the triangles $\triangle WXY$ and $\triangle WZY$. We have $|WX| = |WZ|$ and $|XY| = |ZY|$ because $WXYZ$ is a rhombus, and the side WY is shared; therefore $\triangle WXY \equiv \triangle WZY$ by SSS. Therefore $\angle XWY = \angle ZWY$. Now let the diagonals WY and XZ intersect at the point V. We have $|WX| = |WZ|$ again, and $\angle XWV = \angle ZWV$ (just renaming the angles from before), and trivially $|WV| = |WV|$. Therefore $\triangle XWV \equiv \triangle ZWV$ by SAS. We conclude that $\angle WXV = \angle WXZ$, which shows that WY and XZ are perpendicular, and that $|XV| = |ZV|$, which shows that WY bisects XZ. A similar argument shows that XZ bisects WY as well.

Now the rhombus is made up of four right triangles, each of which has legs of length $|WV| = \frac{1}{2}|WY|$ and $|XV| = \frac{1}{2}|XZ|$ and hypotenuse $|WX|$. Therefore, if we want the rhombus to have side length equal to $|EF|$, we are hoping to have

$$|EF|^2 = \left(\tfrac{1}{2}\right)^2|WY|^2 + \left(\tfrac{1}{2}\right)^2|XZ|^2, \text{ or } |WY|^2 + |XZ|^2 = 4|EF|^2.$$

Moreover, the area of each of the four right triangles is $\frac{1}{2}(\frac{1}{2}|WY|)(\frac{1}{2}|XZ|) = \frac{1}{8}|WY||XZ|$, and so the total area of the rhombus is $4 \cdot \frac{1}{8}|WY||XZ| = \frac{1}{2}|WY||XZ|$. Since we want this to equal the area of the square $ABCD$, we are also hoping that

$$|AB|^2 = \tfrac{1}{2}|WY||XZ|, \text{ or } |WY||XZ| = 2|AB|^2.$$

We are given the values of $|AB|$ and $|EF|$, and we would like to solve this pair of equations for $|WY|$ and $|XZ|$.

There are several ways to do this algebraically, and you can use your favorite one. My preferred method is to note that

$$\begin{aligned}
\big(|WY| + |XZ|\big)^2 &= |WY|^2 + |XZ|^2 + 2|WY||XZ| = 4|EF|^2 + 4|AB|^2, \\
\big(|WY| - |XZ|\big)^2 &= |WY|^2 + |XZ|^2 - 2|WY||XZ| = 4|EF|^2 - 4|AB|^2.
\end{aligned}$$

Therefore $|WY| + |XZ| = 2\sqrt{|EF|^2 + |AB|^2}$ and $|WY| - |XZ| = 2\sqrt{|EF|^2 - |AB|^2}$ (note that the quantity $|EF|^2 - |AB|^2$ is positive by the assumption that $|EF| > |AB|$), and this pair of equations is easy to solve for $|WY|$ and $|XZ|$ by adding and subtracting. We get

$$|WY| = \sqrt{|EF|^2 + |AB|^2} + \sqrt{|EF|^2 - |AB|^2}$$
$$|XZ| = \sqrt{|EF|^2 + |AB|^2} - \sqrt{|EF|^2 - |AB|^2}.$$

Now we can proceed with the actual construction! Using Lemma 3.4.2, construct segments of length $|EF| \cdot |EF| = |EF|^2$ and $|AB| \cdot |AB| = |AB|^2$. Next using Lemma 3.4.1, construct segments of length $|EF|^2 + |AB|^2$ and $|EF|^2 - |AB|^2$. Now using Lemma 3.4.5, construct segments of length $\sqrt{|EF|^2 + |AB|^2}$ and $\sqrt{|EF|^2 - |AB|^2}$. Finally, bisect each of these segments to construct segments of length $\frac{1}{2}\sqrt{|EF|^2 + |AB|^2}$ and $\frac{1}{2}\sqrt{|EF|^2 - |AB|^2}$.

Construct two perpendicular lines intersecting at any constructible point V, and construct the circle of radius $\frac{1}{2}\sqrt{|EF|^2 + |AB|^2}$ centered at V; let it intersect one of the perpendicular lines at T and T' and the other line at U and U'. Now construct circles of radius $\frac{1}{2}\sqrt{|EF|^2 - |AB|^2}$ centered at T, T', U, and U'. For the first line, let W and Y be the points of intersection such that T is between W and V and T' is between Y and V; for the second line, let X and Z be the points of intersection such that X is between U and V and Z is between U' and V. With this construction, we have $|WY| = \sqrt{|EF|^2 + |AB|^2} + \sqrt{|EF|^2 - |AB|^2}$ and $|XZ| = \sqrt{|EF|^2 + |AB|^2} - \sqrt{|EF|^2 - |AB|^2}$ as desired, which makes $|WX| = |XY| = |YZ| = |ZW| = |EF|$ and $|WXYZ| = |AB|^2$ as can be verified algebraically (by the reverse of the above derivation).

[There are other ways to approach this problem. For example, the area of a rhombus $WXYZ$ can be shown to equal $|WX|^2 \sin W$. Therefore, we could construct an angle $\angle A$ such that $\sin A = |AB|^2/|EF|^2$, and then use this angle to build our rhombus with sides of length $|EF|$.]