Math 308, Section 101
Solutions for Homework #5
(due October 27, 2004)

I. Using Theorem 3.6.5, prove that there is no construction that "pentasects" an arbitrary angle. That is, show that there is a constructible angle measuring x° such that it is impossible to construct an angle measuring $(x/5)^\circ$.

We know that we can construct a regular pentagon; in particular, we can construct its central angle of $360^\circ/5 = 72^\circ$. However, Theorem 3.6.5 tells us that we cannot construct a regular 25-sided polygon (since $25 = 5^2$ is a proper prime power). In particular, we cannot construct an angle of measure $360^\circ/25 = 14.4^\circ$. (If we could, we could copy it at the center of some circle 25 times and thus construct the regular 25-gon.) But pentasecting a 72° angle would result in a construction of a 14.4° angle, which we see is impossible. Therefore there is no general construction for pentasecting an arbitrary angle.

[The same type of argument, using Theorem 3.6.5, can prove the following statement: if m has an odd prime factor, then there is no general construction for dividing an arbitrary angle equally into m congruent angles. In other words, we can bisect angles (and re-bisect to 4-sect, re-re-bisect to 8-sect, and so on) with straightedge and compass, but we can only 2^r-sect—we can't m-sect for any other values of m.]

II. Let $\mathcal{C}_A(|AB|)$ be a circle centered at the constructed point A and going through the constructed point B, and let D be any constructed point outside the circle $\mathcal{C}$. Describe, with proof, a construction of a line going through D that is tangent to the circle $\mathcal{C}$.

Construct two perpendicular lines intersecting at a point E. Construct the circle $\mathcal{C}_E(|AB|)$, letting F be one of the points of intersection of this circle with the first line, so that $|EF| = |AB|$. Now construct $\mathcal{C}_F(|AD|)$, letting G be one of the points of intersection of this circle with the second line, so that $|FG| = |AD|$. (Note that $|AD| > |AB|$ since D is outside the original circle $\mathcal{C}_A(|AB|)$, so that this second circle really does intersect the second line.) Since $\triangle EFG$ is a right triangle with hypotenuse $|FG|$, we see by the Pythagorean Theorem that

$$|EG| = \sqrt{|FG|^2 - |EF|^2} = \sqrt{|AD|^2 - |AB|^2}.$$

Now construct $\mathcal{C}_D(|EG|)$, which intersects the original circle $\mathcal{C}_A(|AB|)$ at two points H and H', so that $|DH| = |DH'| = |EG|$ and $|AH| = |AH'| = |AB|$. We claim that the line DH is tangent to the original circle $\mathcal{C}_A(|AB|)$ (as is DH'). To see this, note that

$$|DH|^2 + |HA|^2 = |EG|^2 + |AB|^2 = \left(\sqrt{|AD|^2 - |AB|^2}\right)^2 + |AB|^2 = \left(|AD|^2 - |AB|^2\right) + |AB|^2 = |AD|^2.$$

By the converse of the Pythagorean Theorem, we conclude that $\triangle DHA$ is a right triangle with right angle $\angle DHA$. In other words, the line DH is perpendicular to the radius HA of the original circle $\mathcal{C}_A(|AB|)$. This implies that DH is tangent to the circle, as desired.

[Some students calculated the same length using the Power of the Point Theorem. Others noted that the circle with diameter AD will intersect the original circle $\mathcal{C}_A(|AB|)$ at the point H (can you prove it?), which gives a very quick construction!]

III. Given three distinct, already constructed points A, B, and C, demonstrate that the Euler line of the triangle $\triangle ABC$ can be constructed.

Construct the perpendicular bisectors of two of the triangle's sides, say AB and AC; these perpendicular bisectors intersect at the circumcenter O of $\triangle ABC$, which is thus a constructible point. Similarly, construct the angle bisectors of two of the triangle's angles, say $\angle A$ and $\angle B$; these angle bisectors intersect at the centroid G of $\triangle ABC$, which is thus a constructible point as well. Therefore the line GO, which is precisely the Euler line of $\triangle ABC$, is constructible as well. [Using the intersection of two altitudes to construct the orthocenter H of $\triangle ABC$ in place of one of the other two centers is just as straightforward.]

IV. Prove the formula

$$4 \arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right) = 45^\circ.$$

Using this formula, describe a construction of a segment of length $1/239$ *units that does not use Lemmas 3.4.2 or 3.4.3 (in other words, one that does not rely on multiplication or inversion of already constructed lengths).*

Define x° and y° to be the angles such that $\tan x^\circ = 1/5$ and $\tan y^\circ = 1/239$. Then the formula we are supposed to prove is simply $4x^\circ - y^\circ = 45^\circ$, or $4x^\circ = y^\circ + 45^\circ$. To prove this, it suffices to prove that $\tan(4x^\circ) = \tan(y^\circ + 45^\circ)$. Now

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - (\tan\alpha)(\tan\beta)},$$

from which it follows (by setting $\beta = \alpha$) that

$$\tan(2\alpha) = \frac{2\tan\alpha}{1 - (\tan\alpha)^2}.$$

From this we obtain

$$\tan(2x^\circ) = \frac{2\tan x^\circ}{1 - (\tan x^\circ)^2} = \frac{2 \cdot \frac{1}{5}}{1 - (\frac{1}{5})^2} = \frac{2/5}{24/25} = \frac{5}{12},$$

and therefore

$$\tan(4x^\circ) = \tan(2(2x^\circ)) = \frac{2\tan(2x^\circ)}{1 - (\tan(2x^\circ))^2} = \frac{2 \cdot \frac{5}{12}}{1 - (\frac{5}{12})^2} = \frac{5/6}{119/144} = \frac{120}{119}.$$

On the other hand,

$$\tan(y^\circ + 45^\circ) = \frac{\tan y^\circ + \tan 45^\circ}{1 - (\tan y^\circ)(\tan 45^\circ)} = \frac{\frac{1}{239} + 1}{1 - \frac{1}{239} \cdot 1} = \frac{240/239}{238/239} = \frac{120}{119},$$

and so $\tan(4x^\circ) = \tan(y^\circ + 45^\circ)$ as claimed.

We now use this to construct a segment of length $1/239$. Starting with our base segment OP with $|OP| = 1$, construct the line ℓ that is perpendicular to OP at O. By copying the segment of length 1 five times, construct a point Q on ℓ such that $|OQ| = 5$. Then by the definition of tan, we have $\tan\angle OQP = 1/5$, and so $\angle OQP = x^\circ$ by the definition of x.

Copy the angle $\angle OQP$ three more times around the point Q to make four adjacent copies $\angle OQP$, $\angle PQR$, $\angle RQS$, and $\angle SQT$. Then $\angle OQT = 4\angle OQP = 4x^\circ$. Now construct a 45° angle (by bisecting one of the existing right angles, say) and copy it onto the same side of the ray QO, forming an angle $\angle OQU$ that is contained inside $\angle OQT$. Then $\angle UQT = \angle OQT - \angle OQU = 4x^\circ - 45^\circ = y^\circ$ by the formula proved above. Now that we have the angle y° constructed, we simply transfer this angle back to the ray PO, forming an angle $\angle OPV$, where V is the intersection of the angle's other ray with the perpendicular ℓ. Then $\triangle OPV$ is a right triangle with right angle $\angle O$, and so $|OV| = |OP| \tan \angle OPV = 1 \cdot \tan y^\circ = 1/239$ as desired.

[Note: If we express angle measure in radians rather than degrees, there is a lovely formula (a power series) for $\arctan x$:

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots .$$

Since 45° equals $\pi/4$ radians, the formula to be proved in this problem becomes

$$\begin{aligned}\pi &= 16 \arctan\left(\frac{1}{5}\right) - 4 \arctan\left(\frac{1}{239}\right) \\ &= 16\Big(\frac{1}{5} - \frac{1}{5^3 \cdot 3} + \frac{1}{5^5 \cdot 5} - \frac{1}{5^7 \cdot 7} + \cdots\Big) - 4\Big(\frac{1}{239} - \frac{1}{239^3 \cdot 3} + \frac{1}{239^5 \cdot 5} - \frac{1}{239^7 \cdot 7} + \cdots\Big).\end{aligned}$$

This is actually an excellent formula to use if you want to calculate a lot of digits of π. For example, if you stop at the term $1/(5^{145} \cdot 145)$ in the first infinite sum and the term $-1/(239^{43} \cdot 43)$ in the second infinite sum, the resulting expression matches π to the first 100 decimal places.]

V. Let $d = 2\cos 40^\circ$. *Based on the fact that* $3 \cdot 40^\circ = 120^\circ$ *and that* $\cos 120^\circ$ *is a known quantity, find (with proof) an polynomial* $f(x)$ *with integer coefficients, irreducible over the integers, such that* $f(d) = 0$. *Conclude, without using Theorem 3.6.5, that it is impossible to construct a regular nonagon (nine-sided polygon).*

Recall the triple-angle formula $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$ derived in class. From this we see that

$$8(\cos 40^\circ)^3 - 6\cos 40^\circ = 2\cos(3 \cdot 40^\circ) = 2\cos 120^\circ = 2\big(-\tfrac{1}{2}\big) = -1.$$

If we set $d = 2\cos 40^\circ$, this equation becomes $d^3 - 3d = -1$. In other words, d is a root of the polynomial $f(x) = x^3 - 3x + 1$. Suppose, for the sake of contradiction, that $f(x)$ is not irreducible over the integers. Then $f(x)$ factors as $f(x) = g(x)h(x)$ where both $g(x)$ and $h(x)$ are nonconstant polynomials with integer coefficients. Since $\deg f = 3$, one of the two polynomials $g(x)$ and $h(x)$ must have degree 1 and the other degree 2. In particular, $f(x)$ has a rational root. However, the only possible rational roots of $f(x)$ are ± 1, since the leading coefficient and the constant term of $f(x)$ both equal 1. We quickly check that $f(1) = -1$ and $f(-1) = 3$, and so $f(x)$ does not have a rational root, which is a contradiction. Therefore $f(x)$ is actually irreducible over the integers. Since d is a root of $f(x)$, which has degree 3, we see from Theorem 3.6.4 that d is not a constructible length. Therefore we cannot construct a regular nonagon (if we could, then we could quickly construct a segment of length $2\cos 40^\circ = d$).

[It is not too hard to figure out that the other two roots of the polynomial $f(x)$ are $2\cos 160^\circ = -2\cos 20^\circ$ and $2\cos 280^\circ = 2\cos 80^\circ$.]