Math 308, Section 101
Solutions for Homework #6
(due November 5, 2004)

I. List all of the numbers n between 2,000 and 3,000 for which the regular n-gon can be constructed. Describe briefly why your list is correct.

Between Theorems 3.6.1, 3.6.3, and 3.6.5, we know that a number $n = 2^j m$ (with m odd) has the property that a regular n-gon can be constructed with straightedge and compass if and only if m is 1, a Fermat prime $p = 2^{2^k} + 1$, or a product of distinct Fermat primes. The only Fermat primes below 2,000 are 3, 5, 17, and 257. Therefore there are 16 possible choices for m, and for each one we simply keep multiplying by 2 to see if it has a multiple $2^j m$ between 2,000 and 3,000. (Note that there can be at most one such multiple for any m, since $3{,}000/2{,}000 = 1.5 < 2$.)

- $m = 1$: We find the solution $n = 2^{11} \cdot 1 = 2{,}048$.
- $m = 3$: Since $2^9 \cdot 3 = 1{,}536$ and $2^{10} \cdot 3 = 3{,}072$, this does not yield a solution in the desired range.
- $m = 5$: We find the solution $n = 2^9 \cdot 5 = 2{,}560$.
- $m = 17$: We find $n = 2^7 \cdot 17 = 2{,}176$.
- $m = 257$: We find $n = 2^3 \cdot 257 = 2{,}056$.
- $m = 3 \cdot 5 = 15$: Since $2^7 \cdot 15 = 1{,}920$ and $2^8 \cdot 15 = 3{,}840$, this yields no solution.
- $m = 3 \cdot 17 = 51$: Since $2^5 \cdot 51 = 1{,}632$ and $2^6 \cdot 51 = 3{,}264$, this yields no solution.
- $m = 3 \cdot 257 = 771$: Since $2 \cdot 771 = 1{,}542$ and $2^2 \cdot 771 = 3{,}084$, there's no solution.
- $m = 5 \cdot 17 = 85$: We find $n = 2^5 \cdot 85 = 2{,}720$.
- $m = 5 \cdot 257 = 1{,}285$: We find $n = 2 \cdot 1{,}285 = 2{,}570$.
- $m = 3 \cdot 5 \cdot 17 = 255$: We find $n = 2^3 \cdot 257 = 2{,}040$.
- The other choices $17 \cdot 257$, $3 \cdot 5 \cdot 257$, $3 \cdot 17 \cdot 257$, $5 \cdot 17 \cdot 257$, or $3 \cdot 5 \cdot 17 \cdot 257$ for m are already larger than 3,000, so we have no further solutions.

In summary, the answers are $n = 2{,}040$, 2,048, 2,056, 2,176, 2,560, 2,570, and 2,720.

II. Prove Theorem 3.6.3. You may use the following number theory (Math 312) fact without having to prove it: if m and n are relatively prime, then there exist positive integers x and y such that $mx - ny = 1$.

We are assuming that m and n are relatively prime, and that we can construct the regular m-gon and the regular n-gon; we need to prove that we can construct the regular mn-gon. Start by choosing positive integers x and y such that $mx - ny = 1$. If the vertices of an already constructed regular n-gon, inscribed in a circle with center O, are labeled $P_0, P_1, P_2, \ldots, P_{n-1}$, then consider the angle $\angle P_0 O P_x$, which is certainly constructible. The measure of this angle is simply x times the measure of one central angle of the regular n-gon, or in other words $x \cdot (360°/n)$. In the same way, we can show that an angle of measure $y \cdot (360°/m)$ can be constructed using the already constructed regular m-gon.

Now copy both angles onto the same line—say $\angle BAD = x \cdot (360°/n)$ and $\angle BAC = y \cdot (360°/m)$. What is the measure of the angle $\angle DAC$ thus constructed? It is

$$\angle DAC = \angle BAD - \angle BAC = x \cdot \frac{360°}{n} - y \cdot \frac{360°}{m} = \frac{360(xm - yn)°}{mn} = \frac{360 \cdot 1°}{mn} = \frac{360°}{mn}.$$

In other words, the angle $(360/mn)^\circ$ can be constructed. From this, as we know, we can construct the entire regular mn-gon, as claimed.

III. Let $\angle PQA$ be a right angle, and let ε be any positive number. Show that there exists a point R on the ray QA such that $\angle PRQ < \varepsilon^\circ$.

(Refer to Figure 6.10 on page 125 of Baragar's book.) Begin by choosing a point R_1 on the ray QA such that $|QR_1| = |QP|$. By *pons asinorum*, we know that $\angle QR_1P = \angle QPR_1$. We also know that the sum of the three angles in $\triangle PQR_1$ is less than 180°, and so $\angle PQR_1 + \angle QR_1P + \angle QPR_1 = 90^\circ + 2\angle QR_1P < 180^\circ$. We conclude that $\angle QR_1P < (90/2)^\circ$. Next, choose a point R_2 on the line QA such that R_1 is between R_2 and Q and $|R_1R_2| = |R_1P|$. Notice that $180^\circ - \angle PR_1R_2 = \angle PR_1Q < (90/2)^\circ$. Again, by *pons asinorum* we have $\angle PR_2R_1 = \angle R_1PR_2$, and so the sum of the angles of $\triangle PR_1R_2$ gives us the inequality $\angle PR_1R_2+\angle PR_2R_1+\angle R_1PR_2 = \angle PR_1R_2+2\angle PR_2R_1 < 180^\circ$, or $2\angle PR_2R_1 < 180^\circ - \angle PR_1R_2 < (90/2)^\circ$. We conclude that $\angle PR_2R_1 < (90/2^2)^\circ$.

Continue recursively in this way: once the point R_n has been chosen, and it has been shown inductively that $\angle PR_nR_{n-1} < (90/2^n)^\circ$, choose a point R_{n+1} on the line QA such that R_n is between R_{n+1} and Q and $|R_nR_{n+1}| = |R_nP|$. Notice that $180^\circ - \angle PR_nR_{n+1} = \angle PR_nQ < (90/2^n)^\circ$. Again, by *pons asinorum* we have $\angle PR_{n+1}R_n = \angle R_nPR_{n+1}$, and so the sum of the angles of $\triangle PR_nR_{n+1}$ gives us the inequality $\angle PR_nR_{n+1} + \angle PR_{n+1}R_n + \angle R_nPR_{n+1} = \angle PR_nR_{n+1} + 2\angle PR_{n+1}R_n < 180^\circ$, or $2\angle PR_{n+1}R_n < 180^\circ - \angle PR_nR_{n+1} < (90/2^n)^\circ$. We conclude that $\angle PR_{n+1}R_n < (90/2^{n+1})^\circ$. (This can be written as a formal induction, but I hope the argument is clear.)

All it remains to note is that when n is large enough, $90/2^n$ is smaller than any predetermined positive number ε. (In fact, this will be the case as soon as $n > \log_2(90/\varepsilon)$.) Since the angle $\angle PR_nR_{n-1}$ is simply the angle $\angle PR_nQ$, we see that once n is large enough, the angle $\angle PR_nQ$ will have measure smaller than ε°, as desired.

IV. (Baragar, p. 125, #6.8) Prove that if alternate interior angles of a transversal ℓ to two lines ℓ_1 and ℓ_2 are equal, then ℓ_1 and ℓ_2 are ultraparallel.

Let the points of intersection of ℓ with ℓ_1 and ℓ_2 be A and B, respectively. Suppose that the lines ℓ_1 and ℓ_2 intersected at a point C. Then the two angles $\angle CAB$ and $\angle CBA$ would add to 180° (one of them is one of the alternate interior angles, and the other is the supplementary angle of the other interior angle), and that would make the sum of the angles in the triangle $\triangle ABC$ greater than 180°, which is impossible. Therefore ℓ_1 and ℓ_2 cannot intersect. It remains to show that they cannot be parallel either.

By the Fact proven in class, any two lines that have a transversal resulting in equal alternate interior angles also have a common perpendicular. So let D and E be points on ℓ_1 and ℓ_2, respectively, such that DE is perpendicular to both lines. Now suppose that ℓ_1 and ℓ_2 were parallel. Then since DE is perpendicular to ℓ_2, the angle of parallelism at D with respect to the line ℓ_2 would be (by definition) the angle between DE and the parallel line ℓ_1, which is 90°. But this is impossible, since angles of parallelism are always strictly less than 90°. Therefore ℓ_1 and ℓ_2 can neither intersect nor be parallel. We conclude that they must be ultraparallel.

V. Suppose you have a Euclidean circle centered at O with two perpendicular diameters MN and AB. Let P be a point on the segment AB other than O, and let α° be the measure of the (Euclidean) angle $\angle PMN$. Now think of the circle as forming the boundary of the hyperbolic plane, so that the diameter MN is a hyperbolic-line ℓ and O and P are points in the hyperbolic plane. Prove that the angle of parallelism $\Pi(|PO|)$ has the value $(90 - 2\alpha)^\circ$.

We first prove a simple lemma: if Q is a point outside a circle centered at C, and N and P are the two points on the circle such that QN and QP are tangent to the circle, then $|QN| = |QP|$. To see this, note that $\angle CNQ = 90^\circ$ since QN is the tangent line, and so by the Pythagorean Theorem, $|QN| = \sqrt{|CQ|^2 - |CN|^2}$. By the same argument, $|QP| = \sqrt{|CQ|^2 - |CP|^2}$. But $|CN| = |CP|$ since both segments are radii, and so the expressions for $|QN|$ and $|QP|$ are equal.

Now consider the hyperbolic plane with two perpendicular Euclidean-diameters MN and AB drawn, intersecting at O. Let P be a point above O. How do we draw the angle of parallelism at P with respect to the horizontal line? We have to draw the two lines through P that are parallel to the horizontal line; this means that they should be arcs of Euclidean-circles that are tangent to MN at the points M and N, as in Figure 1. If we draw the Euclidean-tangent-lines to these Euclidean-circles at P (say one of them intersects the horizontal line at Q, then the angle of parallelism we are looking for is precisely the Euclidean-angle $\angle OPQ$, which is the angle between the hyperbolic-lines OA and PM.

Now let's focus on a piece of this diagram, as shown in Figure 2, and consider it as a Euclidean picture. We are given that $\angle MPO = \alpha^\circ$, and we are trying to determine $\beta^\circ = \angle OPQ$. Since MO and OP are perpendicular, we have $\angle MOP = 90^\circ$ and hence $\angle MPO = 180^\circ - \angle MOP - \angle MPO = (90 - \alpha)^\circ$. On the other hand, the segments QM and QP are tangent to the same circle; by the lemma we started with, we see that $|QM| = |QP|$. Therefore by *pons asinorum*, $\angle QPM = \angle QMP = \alpha^\circ$. We conclude that $\beta^\circ = \angle OPQ = \angle OPM - \angle QPM = (90 - \alpha)^\circ - \alpha^\circ = (90 - 2\alpha)^\circ$, as claimed.

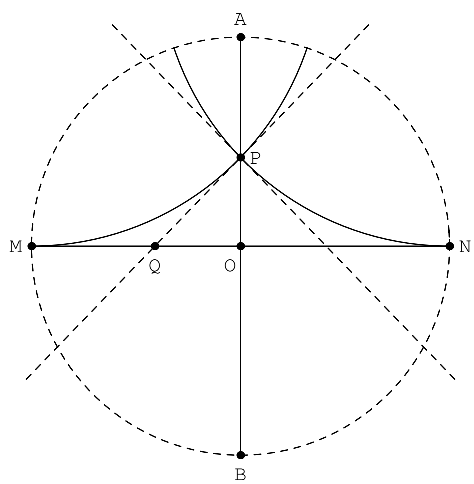

Figure 1

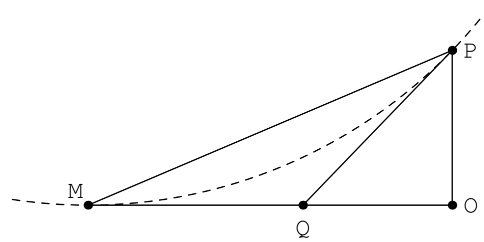

Figure 2

VI. Using the disk model of the hyperbolic plane, draw a triangle $\triangle ABC$ with three $45°$ angles, where A is the Euclidean-center of the Euclidean-circle that forms the boundary of the hyperbolic plane. If O is the Euclidean-center of the Euclidean-circle that defines the line BC, prove that $ABOC$ is a Euclidean-rhombus. Now consider the isometry f that is the reflection of the hyperbolic plane in the line BC. Draw the images of the lines AB and AC under f and their intersection A'. What are the measures of the angles $\angle ABA'$ and $\angle ACA'$? Conclude that the Euclidean-center of the Euclidean-circle defining the line $A'B$ lies on the Euclidean-line AB.

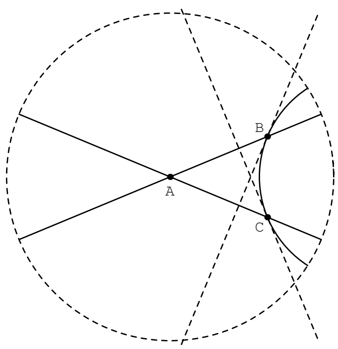

Figure 1

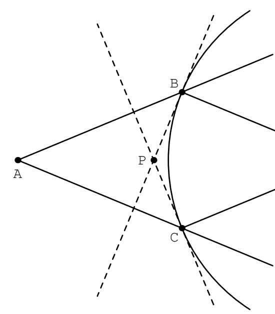

Figure 2

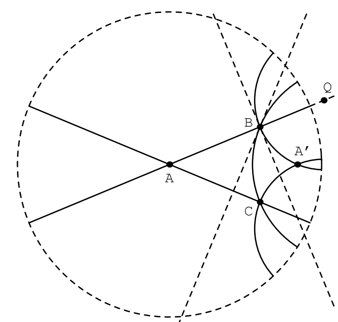

Figure 3

We know that the lines AB and AC must be Euclidean-diameters that intersect at a $45°$ angle at A. We also know that the line BC must be an arc of a Euclidean-circle, and that the Euclidean-tangent-line to the circle at B must meet the line AB in a $45°$ angle, and similarly at C. Therefore the $45°$-$45°$-$45°$ triangle $\triangle ABC$ will appear as in Figure 1. To investigate the Euclidean quadrilateral $ABOC$, we begin by noting that $|OB| = |OC|$ since both segments are radii of the same Euclidean-circle. Now label by P the point of intersection of the two Euclidean-tangent-lines at B and C, as in Figure 2. We have already mentioned that $\angle BAC = \angle ABP = \angle ACP = 45°$. On the other hand, $\angle OBP = 90° = \angle OCP$ since the radii are always perpendicular to the tangent lines. Therefore $\angle ABO = \angle ABP + \angle OBP = 45° + 90° = 135°$, and similarly $\angle ACO = 135°$. Since $\angle CAB + \angle ABO = 45° + 135° = 180°$, we conclude that the Euclidean-lines AC and BO are parallel; similarly, the fact that $\angle BAC + \angle ACO = 180°$ shows that the Euclidean-lines AB and CO are parallel. This means that $ABOC$ is a parallelogram, and hence the opposite sides are equal, that is, $|AC| = |BO|$ and $|AB| = |CO|$. Since we already know that $|OB| = |OC|$, we have shown that all four sides are equal, and so $ABOC$ is a rhombus.

Returning to the hyperbolic plane: suppose we look at the reflection f in the line BC, which clearly satisfies $f(B) = B$ and $f(C) = C$. Reflections (all isometries) preserve angles, and so $f(A) = A'$, it must be the case that $\angle ABC = \angle A'BC$. Thus $\angle A'BC$ must be $45°$, and so $\angle ABA' = \angle ABC + \angle A'BC = 45° + 45° = 90°$. Therefore, the reflection of the line AB will be a new line $A'B$, which is an arc of a Euclidean-circle passing through B in such a way that the Euclidean-tangent-line at B is perpendicular to the line AB. The result looks like Figure 3 (similar remarks hold for what happens at C, so that $\angle ACA' = 90°$ in particular). Now the Euclidean-line AB intersects the Euclidean-circle through B and A' in Figure 3 perpendicular to the Euclidean-tangent-line at B, and so we know that AB actually passes through the Euclidean-center Q of the Euclidean-circle in question.