

Math 308, Section 101
Solutions for Homework #7
 (due November 22, 2004)

I. How do the axioms for hyperbolic geometry differ from the axioms for Euclidean geometry? For any axiom that is different, write down the hyperbolic-geometry version of the axiom.

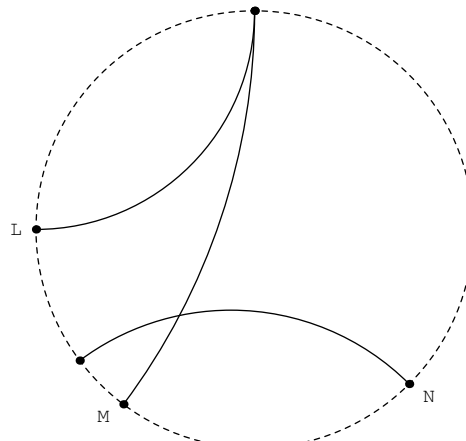
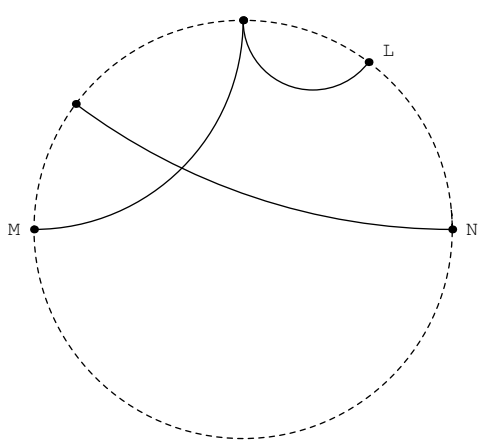
There is only one difference in the axioms, namely in Axiom 5. In place of the Euclidean version (the Parallel Postulate), hyperbolic geometry has the following version of Axiom 5: Given any line ℓ and any point P not on ℓ , there are at least two distinct lines through P that do not intersect ℓ .

II. Let D_M denote the diameter of the moon, D_S the diameter of the sun, R_M the distance from the earth to the moon, and R_S the distance from the earth to the sun. Explain, including a diagram, how a solar eclipse (when the moon passes in front of the sun, as seen from the earth) gives an equation relating these four quantities.

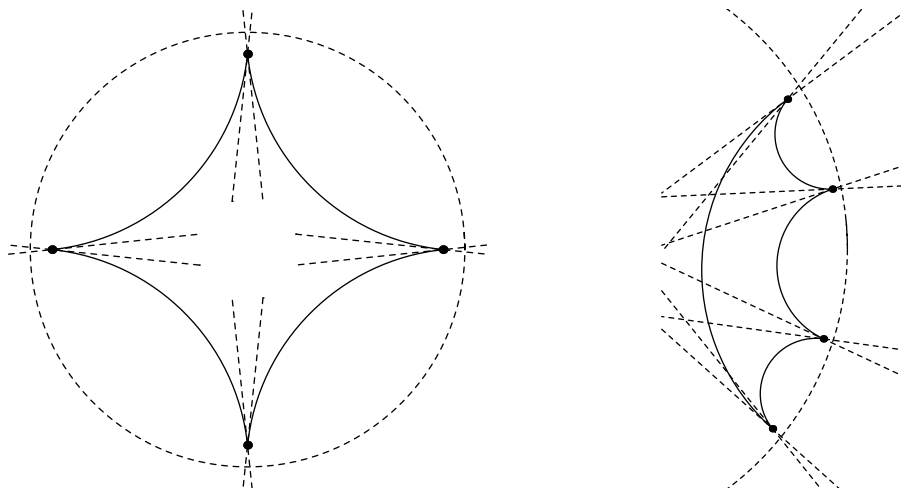
This problem is discussed in Chapter 2 of the textbook. (I apologize for not having time to draw up a nice diagram for this solution set.)

III. For this question, “draw” means “draw, with reasonable accuracy, a picture using the disk model of the hyperbolic plane”. For this question, do not allow any of your hyperbolic lines to look like diameters of the boundary Euclidean circle.

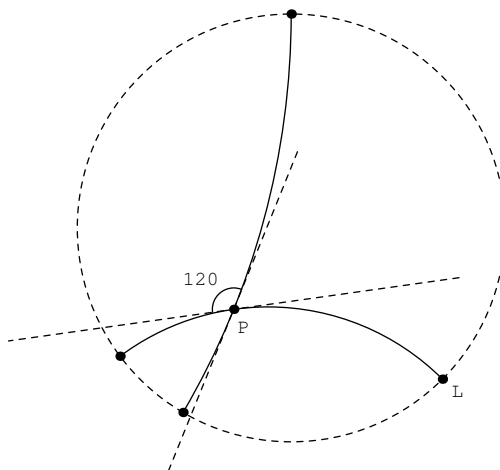
- (a) Draw three lines \mathcal{L} , \mathcal{M} , and \mathcal{N} such that \mathcal{L} and \mathcal{M} are parallel to each other, \mathcal{L} and \mathcal{N} are ultraparallel to each other, and \mathcal{M} and \mathcal{N} intersect each other. (Be sure to label the lines correctly.)
- (b) Draw a quadrilateral $ABCD$ such that each of its four angles is less than 30° . (Be sure that all four vertices really are points in the hyperbolic plane.)
- (c) Draw a line \mathcal{L} through a point P , and draw the image of \mathcal{L} under the isometry (of the hyperbolic plane) that is a rotation around P through a 120° angle. Mark the 120° angle in your drawing, and label the line \mathcal{L} to distinguish it from its image line.



Two equally good solutions to part (a). Note that the lines \mathcal{L} and \mathcal{M} are represented by arcs of Euclidean-circles that are tangent at the boundary.



Two equally good solutions to part (b). It is important that the vertices of the quadrilateral do not lie on the boundary circle. The tangent lines of the various arcs illustrate the angles at each of the vertices.



A solution to part (c). Note that the only way to detect the angle of rotation is to look at the tangent lines; there's no good way to know in advance what the image line will look like, but it's easier to spot once it's drawn.

IV. One of the two angles 2° and 3° is constructible using a straightedge and compass, while the other one is not. Which one is which? Explain your answers.

A 3° angle is constructible (with straightedge and compass), but a 2° angle is not constructible.

Since $360^\circ/120 = 3^\circ$, a 3° angle is constructible if and only if a regular 120-gon is constructible. We know from class that a regular octagon (8-gon) and a regular 15-gon are both constructible. Since $\gcd(8, 15) = 1$ —that is, 8 and 15 are relatively prime—Theorem 3.6.3 tells us that a regular (8×15) -gon is also constructible. Since $8 = 120$, this shows that a 3° angle can be constructed.

Since $360^\circ/180 = 2^\circ$, a 2° angle is constructible if and only if a regular 180-gon is constructible. Note that if a regular 180-gon were constructible, then by taking every 20th

vertex, we would find a constructed regular 9-gon. However, we know that a regular 9-gon is not constructible, and therefore a regular 180-gon cannot be constructed either. We conclude that a 2° angle cannot be constructed. [Note: understanding why this half of the argument works is important. It has nothing to do with anything being relatively prime.]

V. Let A and B be already constructed points, and let C be an already constructed circle that is centered at A . Suppose that the point B is inside the circle C . Describe how to construct, with straightedge and compass, a circle that goes through B and is tangent to C .

Construct the line through A and B . This yields two points of intersection with the already constructed circle C ; call one of the points of intersection D . By constructing the perpendicular bisector of the segment BD , construct the midpoint M of BD , so that $|MB| = |MD|$. Finally, construct $C' = C_M(|MB|)$, which clearly goes through B and, since $|MB| = |MD|$, through D is well. Therefore the two circles C and C' intersect at D . Moreover, the line joining the centers A and M of the two circles goes through this point of intersection, and so it is the only point of intersection. (This fact can also be established from scratch using the “equality” case of the Triangle Inequality.) This shows that C' , which goes through B , is indeed tangent to C .

[Note 1: There are actually lots of circles C' with the desired property, but this is the easiest one to describe. In fact the above construction yields two such circles, depending on which point of intersection is labeled D at the first step—it does not have to be the one closer to B .]

[Note 2: in some versions of the midterm, the points A and B were named P and Q , and the point Q was outside (rather than inside) the circle. This does not change the proof in any significant way, only the picture drawn to illustrate the proof.]

VI. Describe how to construct, with straightedge and compass, an angle such that

$$\sin \quad = \frac{\sqrt{3} + \sqrt{5}}{7}.$$

You may use the fact that $0 < \sqrt{3} + \sqrt{5} < 7$ without having to prove it. You do not have to describe each individual line and circle drawn, as long as you indicate the steps you are taking and why they are possible.

We know that given any positive rational number, a segment can be constructed whose length is the given number. In particular, we can construct segments of lengths 3, 5, and 7. We also know that we can construct a segment whose length is the square root of the length of any already constructed segment; in particular, we can now construct segments of length $\sqrt{3}$ and $\sqrt{5}$. Moreover, we can construct a segment whose length is the sum of the lengths of two already constructed segments; therefore we can construct a segment of length $\sqrt{3} + \sqrt{5}$.

Now construct two perpendicular lines intersecting at a point A . Construct $C_A(\sqrt{3} + \sqrt{5})$, and let B be a point of intersection of this circle and the vertical line through A . Next construct $C_B(7)$, and let C be a point of intersection of this circle and the horizontal line through A . (The fact that $7 > \sqrt{3} + \sqrt{5}$ ensures that this circle actually intersects the line.) Then $\triangle ABC$ is a right triangle with $\angle A = 90^\circ$, and so by the definition of sine we have $\sin \angle ACB = |AB|/|BC| = (\sqrt{3} + \sqrt{5})/7$. Therefore $\angle ACB =$ is the desired angle.

[Note: in some versions of the midterm, the angle needed to satisfy $\cos = (\sqrt{3} + \sqrt{7})/5$. The solution for this version is only superficially different.]

VII. *In hyperbolic geometry, suppose that the line \mathcal{L} intersects two other lines \mathcal{M} and \mathcal{N} in such a way that the two alternate interior angles are equal. Prove that the lines \mathcal{M} and \mathcal{N} are ultraparallel.*

This was problem IV of Homework #6, so the solution has already been posted. Note also that knowing what we know now about singly asymptotic triangles, we can prove that \mathcal{M} and \mathcal{N} cannot be parallel another way: if they were, then they would form a singly asymptotic triangle whose two angles sum to 180° , which is a contradiction.