Math 308, Section 101
Solutions for Homework #8
(due December 3, 2004)

I. (Baragar, p. 110, #5.11) How many square faces are on the snub cube?

By definition, the snub cube is represented by the sequence $(3, 3, 3, 3, 4)$; that is, there are four (equilateral) triangles and one square at every vertex. Let F_3 represent the number of triangular faces on the snub cube and F_4 the number of square faces. We use the Euler characteristic equation $F - E + V = 2$ to determine all of the values.

Since every square has four vertices, and since every vertex is attached to exactly one square, we immediately get the equation $V = 4F_4$. Every triangle has three vertices, but now every vertex has four triangle vertices attached to it; therefore $4V = 3F_3$ as well, which gives the relationship $16F_4 = 3F_3$. Also, each triangle has three edges and each square four, but we count each edge twice in this way; therefore $2E = 3F_3 + 4F_4$. Since $3F_3 = 16F_4$, we can write this last equation as $2E = 16F_4 + 4F_4$, or $E = 10F_4$. Now we can express everything in terms of F_4, once we note that $F = F_3 + F_4$ and $F_3 = \frac{16}{3}F_4$:

$$\begin{aligned} V - E + F &= 2 \\ 4F_4 - 10F_4 + (F_3 + F_4) &= 2 \\ 4F_4 - 10F_4 + \frac{16F_4}{3} + F_4 &= 2 \\ \frac{F_4}{3} &= 2 \\ F_4 &= 6. \end{aligned}$$

Therefore the snub cube has six square faces. (We also quickly see that it has 32 triangular faces, 24 vertices, and 60 edges.)

II. Calculate the dihedral angle of a regular tetrahedron (that is, the angle between two of its faces).

For arithmetical simplicity later, we let the edge length of our regular tetrahedron equal 2. To find the dihedral angle, we look at the tetrahedron from an angle at which one of the edges is aligned straight away from us, so that it looks like a single point. The resulting image is a triangle, where the side opposite that point is an edge of the tetrahedron (and thus has length 2), and the two sides adjacent to that point are altitudes of faces of the tetrahedron. Since each face is an equilateral triangle of side-length 2, the altitudes have length $\sqrt{3}$. Moreover, the vertex angle ϕ of this triangle is precisely the dihedral angle we want to measure.

Therefore, ϕ is the vertex angle of an isosceles triangle with base 2 and sides $\sqrt{3}$. Solving the Law of Cosines for $\cos\phi$, we get

$$\cos\phi = \frac{a^2 + b^2 - c^2}{2ab} = \frac{\sqrt{3}^2 + \sqrt{3}^2 - 2^2}{2\sqrt{3}\sqrt{3}} = \frac{3 + 3 - 4}{6} = \frac{1}{3}.$$

In other words, $\phi = \arccos\frac{1}{3}$. As it happens, this angle is about $70.5°$.

(Recall that the dihedral angle of the regular octahedron is $\arccos(-\frac{1}{3})$, which is exactly the supplemental angle to ϕ. This means that if you have regular tetrahedra and a regular octahedron of the same edge length, then the tetrahedra will slide neatly under the corners of the octahedron on the tabletop. In fact, if you start with a big regular tetrahedron and cut off its four corner tetrahedra, the result will be a regular octahedron.)

III. (Baragar, p. 149, #7.34) In the upper hand plane model $\mathcal{H}$, carefully draw the asymptotic triangle with vertices i, $1+i$, and 1. Is the map $\gamma = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$ an isometry of $\mathcal{H}$? In the same diagram, carefully draw the image of the asymptotic triangle under the action of γ.

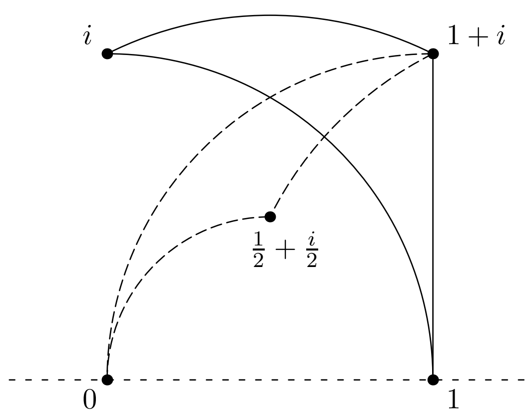
The problem is referring to the map T_γ, of course, and in this case $T_\gamma(z) = (z-1)/z = 1 - 1/z$. It is an isometry since γ is in $SL_2(\mathbb{R})$ (note that $\det \gamma = 1 \cdot 0 - (-1) \cdot 1 = 1$). We quickly compute that

$$T_\gamma(i) = 1 - \frac{1}{i} = 1 - (-i) = 1 + i$$

$$T_\gamma(1+i) = 1 - \frac{1}{1+i} = 1 - \frac{1-i}{(1+i)(1-i)} = 1 - \frac{1-i}{2} = \tfrac{1}{2} + \tfrac{1}{2}i$$

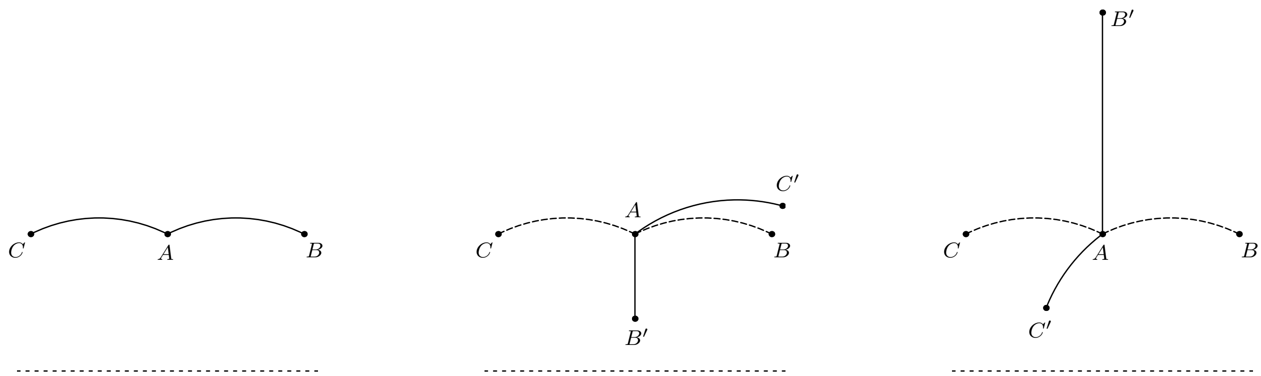
$$T_\gamma(1) = 1 - \frac{1}{1} = 0.$$

All of the circular arcs making up the hyperbolic segments are easily determined (their centers are 0, $\frac{1}{2}$, 1, and $1\frac{1}{2}$), and the result is the following diagram.



IV. Let $A = i$, $B = 1+i$, and $C = -1+i$. Suppose that f is a direct isometry of the Poincaré half-plane model of hyperbolic geometry such that $f(A) = A$ and $f(B)$ is on the vertical line going through A. If $f(B)$ is above A, is the real part of $f(C)$ positive, negative, or zero? What if $f(B)$ is below A?

We don't have to figure out the exact measure of $\angle BAC$ to answer this question (although as it happens, $\angle BAC$ is a bit less than $127°$). We only need to observe that $\angle BAC$, when traveling counterclockwise from the ray AB to the ray AC, is less than $180°$ (notice that the tangent lines to the arcs at A both rise above the horizontal line through A). Let $B' = f(B)$ and $C' = f(C)$; we are given that $f(A) = A$. Since f is a direct isometry, that means that $\angle B'AC'$, when traveling *counterclockwise* from the ray AB' to the ray AC', is less than $180°$. (If f had been an orientation-reversing isometry, the image angle would be oriented clockwise.) Therefore, if $f(B) = B'$ is below A, then the tangent line to the circular arc AC' at A must have a positive slope; this means that the circular arc representing the ray AC' goes to the right of the point A, which is to say that $f(C) = C'$ has positive real part. If on the other hand, $f(B) = B'$ is above A, then $f(C) = C'$ will have negative real part.



V. Write the fractional linear transformation $f(z) = -1/(4z+10)$ as a composition using the functions $g(z) = -1/z$ and $h_\alpha(z) = z+\alpha$. You may use g and h multiple times in the composition, and you may use various values for α.

Our first thought might be to write down $\gamma = \begin{bmatrix} 0 & -1 \\ 4 & 10 \end{bmatrix}$, but this matrix is not in $SL_2(\mathbb{R})$, as its determinant is 4. However, we can fix this by dividing through by the square root of the determinant. Therefore we define $\gamma = \begin{bmatrix} 0 & -1/2 \\ 2 & 5 \end{bmatrix}$, which is in $SL_2(\mathbb{R})$.

Now we look to the proof of Theorem 7.6.4, noticing that the "a" in our matrix is 0, so we have to read the proof's last paragraph. If we set, as usual,

$$\sigma = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \tau_\alpha = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix},$$

then we are led to notice that

$$\sigma\gamma = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1/2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} -2 & -5 \\ 0 & -1/2 \end{bmatrix}.$$

At this point, the formula from earlier in the proof (mentioned in class as well) is relevant, namely

$$\sigma\tau_{-(1+c)/a}\sigma\tau_{-a}\sigma\tau_{(b-1)/a} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

We want to obtain the matrix $\sigma\gamma$, and so we have $-(1+c)/a = -(1+0)/(-2) = 1/2$and $-a = 2$ and $(b-1)/a = (-5-1)/(-2) = 3$.

In summary, we have derived the matrix product $\sigma\tau_{1/2}\sigma\tau_2\sigma\tau_3 = \sigma\gamma$. Since $-\sigma\sigma$ is the identity matrix, we can multiply both sides by $-\sigma$ on the left to simplify the equation to $\tau_{1/2}\sigma\tau_2\sigma\tau_3 = \gamma$. Finally, since the product of matrices corresponds to the composition of the appropriate fractional linear transformations, we see that using the five functions

$$h_{1/2}(z) = z + \tfrac{1}{2}, \quad g(z) = -\tfrac{1}{z}, \quad h_2(z) = z+2, \quad g(z) = -\tfrac{1}{z}, \quad \text{and } h_3(z) = z+3,$$

we have $h_{1/2} \circ g \circ h_2 \circ g \circ h_3 = f$. This says, by the way, that

$$\left(\frac{-1}{\left(\frac{-1}{z+3}\right) + 2} \right) + \tfrac{1}{2} = \frac{-1}{4z+10}.$$

(Remark: there are many possible answers to this question.)

VI. Suppose γ is a matrix in $SL_2(\mathbb{R})$ such that the fractional linear transformation T_γ fixes the point i. Show that

$$\gamma = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

for some real number θ.

We write $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, where a, b, c, d are real numbers with $ad - bc = 1$. If T_γ is to fix the point i, then we must have $(ai + b)/(ci + d) = i$ or, cross-multiplying, $ai + b = (ci + d)i = -c + di$. This happens if and only if $d = a$ and $c = -b$, which turns the determinant equation into $a(a) - b(-b) = a^2 + b^2 = 1$. In particular, a is at most 1 in absolute value, so we are free to define $\theta = \arccos a$. Then $a = \cos\theta$, and b equals $\sqrt{1 - a^2} = \sqrt{1 - \cos^2\theta}$, which is either $\sin\theta$ or $-\sin\theta$. Suppose first that $b = \sin\theta$, then we immediately have $c = -b = -\sin\theta$ and $d = a = \cos\theta$, and our matrix is in the desired form. If on the other hand $b = -\sin\theta$, then we simply set $\theta' = -\theta$, so that $a = \cos\theta = \cos(-\theta) = \cos\theta'$ and $b = -\sin\theta = \sin(-\theta) = \sin\theta'$. In this case, we have our matrix in the desired form with θ' instead of θ.

VII. What is the shortest path between the points $A = 12i$ and $B = 7 + 5i$ in the Poincaré half-plane? (Be specific.) Find an isometry that sends A and B to two points that lie on the same vertical line, and use this isometry to calculate the (hyperbolic) distance between A and B. By using the cross ratio with the points A and B directly, calculate the distance between them a second way.

We know that the shortest path between the points $A = 12i$ and $B = 7+5i$ in the Poincaré half-plane is an arc of a Euclidean circle centered on the x-axis. Therefore we let the center of this circle be $(x, 0)$ and solve for the value of x that provides equal distances to both A and B:

$$\begin{aligned} (x-0)^2 + (0-12)^2 &= (x-7)^2 + (0-5)^2 \\ x^2 + 144 &= x^2 - 14x + 49 + 25 \\ 14x &= 49 + 25 - 144 = -70 \\ x &= -5 \end{aligned}$$

Thus the semicircle is centered at the point -5 and has radius $\sqrt{(-5-0)^2 + (0-12)^2} = 13$, so that its endpoints are $-5 - 13 = -18$ and $-5 + 13 = 8$.

We now find an isometry that sends A to B to vertically-aligned points. Following our general recipe, we begin by horizontally translating so that one endpoint, say 8, is sent to 0. That is, we begin with $h_{-8}(z) = z - 8$. Then we invert in the unit circle, using $f(z) = 1/\bar{z}$. The first isometry $h_{-8}(z)$ sends $12i$ to $-8 + 12i$ and $7 + 5i$ to $-1 + 5i$, and so the composed isometry $f \circ h(z) = 1/(\overline{z - 8})$ sends $12i$ and $7 + 5i$ to

$$\frac{-8 + 12i}{(-8)^2 + 12^2} = \frac{-8 + 12i}{208} = -\frac{1}{26} + \frac{3}{52}i \quad \text{and} \quad \frac{-1 + 5i}{(-1)^2 + 5^2} = \frac{-1 + 5i}{26} = -\frac{1}{26} + \frac{5}{26}i.$$

Therefore the distance between A and B is equal to the distance between $-\frac{1}{26} + \frac{3}{52}i$ and $-\frac{1}{26} + \frac{5}{26}i$, which allows us to use the vertical-line formula: the answer is $|\ln\left(\frac{3}{52}/\frac{5}{26}\right)| = |\ln\left(\frac{3}{52} \cdot \frac{26}{5}\right)| = |\ln\frac{3}{10}| = \ln\frac{10}{3}$.

Alternatively, we can use the cross ratio with the points A and B and the endpoints -18 and 8 directly:

$$\begin{aligned}(12i, 7+5i; -18, 8) &= \Big(\frac{12i-(-18)}{12i-8}\Big)\Big/\Big(\frac{7+5i-(-18)}{7+5i-8}\Big)\\ &= \Big(\frac{18+12i}{-8+12i}\cdot\frac{-1+5i}{25+5i}\Big) = \frac{6}{4\cdot 5}\Big(\frac{3+2i}{-2+3i}\cdot\frac{-1+5i}{5+i}\Big)\\ &= \frac{3}{10}\cdot\frac{-3+15i-2i+10i^2}{-10-2i+15i+3i^2} = \frac{3}{10}\cdot\frac{-13+13i}{-13+13i} = \frac{3}{10}.\end{aligned}$$

Therefore the distance from A to B is $|\ln(12i, 7+5i; -18, 8)| = |\ln\frac{3}{10}| = \ln\frac{10}{3}$, as before.

VIII. Consider two points $A = x_1 + i$ and $B = x_2 + i$ in the Poincaré half-plane. Let ℓ_A and ℓ_B be the vertical lines going through A and B, respectively. Show that the angle between the (hyperbolic) lines ℓ_A and AB is

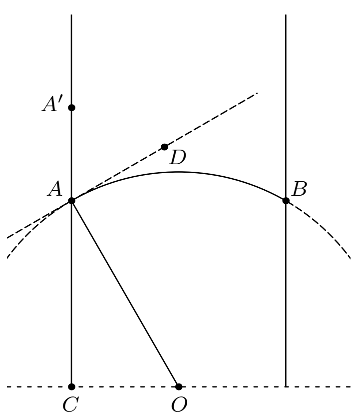
$$\arctan\left(\frac{2}{|x_1 - x_2|}\right),$$

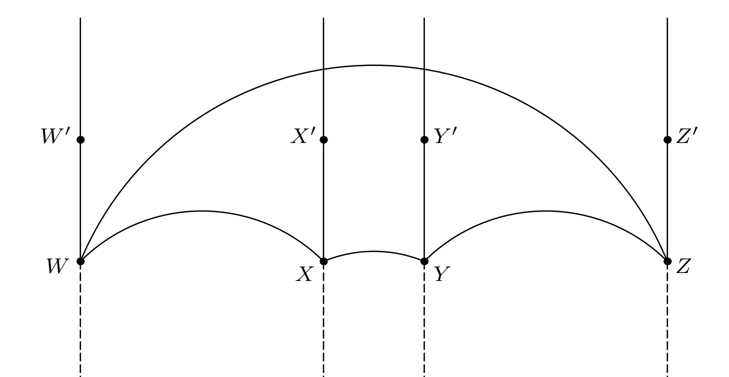
and the same for the angle between the lines ℓ_B and AB. Then show that the (hyperbolic) area of the (hyperbolic) quadrilateral whose vertices are

$$(-1-\sqrt{2})+i, \quad (1-\sqrt{2})+i, \quad (-1+\sqrt{2})+i, \quad \textit{and } (1+\sqrt{2})+i$$

equals $\pi/2$. You may use the values $\tan(\pi/8) = \sqrt{2}-1$ and $\tan(3\pi/8) = \sqrt{2}+1$ without having to prove them.

Let A' be a point on ℓ_A above A, and let $C = x_1$ be the endpoint of ℓ_A. We know that the center of the Euclidean semicircle defining the line AB must lie on the perpendicular bisector of the Euclidean segment AB, and so that center is clearly $O = (x_1+x_2)/2$. Draw the tangent line to this semicircle at A, and let D be a point on that tangent line above A. We are trying to calculate $\angle A'AD$. However, $\angle A'AD = 180° - \angle DAO - \angle OAC = 90° - \angle OAC$, since the Euclidean tangent line AD is perpendicular to the Euclidean radius OA. Also, $\angle AOC = 180° - \angle OCA - \angle OAC = 90° - \angle OAC$ since the sum of the angles of the Euclidean triangle $\triangle ACO$ equals 180°. We conclude that $\angle A'AD = \angle AOC$. However, we can easily read off from the right triangle $\triangle ACO$ that $\tan AOC = |AC|/|CO| = 1/(|x_1 - x_2|/2) = 2/|x_1 - x_2|$. Thus $\angle A'AD = \arctan(2/|x_1 - x_2|)$ as claimed. The same proof applies to the angle at B as well.





Now we calculate the area of the quadrilateral with vertices $W = (-1-\sqrt{2})+i$, $X = (1-\sqrt{2})+i$, $Y = (-1+\sqrt{2})+i$, and $Z = (1+\sqrt{2})+i$. Corollary 7.13.3 tells us that the area of a triangle is π minus the sum of its three angles (measured in radians); and we have seen that we can always divide a quadrilateral into two triangles. It follows easily that the area of a quadrilateral is 2π minus the sum of its four angles. If we let W', X', Y', Z' be points on the vertical lines above W, X, Y, Z respectively, then we have

$$\begin{aligned}
\operatorname{area}(WXYZ) &= 2\pi - \angle ZWX - \angle WXY - \angle XYZ - \angle YZW \\
&= 2\pi - (\angle W'WX - \angle W'WZ) - (\angle X'XW + \angle X'XY) \\
&\qquad - (\angle Y'YX + \angle Y'YZ) - (\angle Z'ZY - \angle Z'ZW) \\
&= 2\pi + 2\angle W'WZ - 4\angle W'WX - 2\angle X'XY,
\end{aligned}$$

where we have taken the symmetries of the various angles into account. By the first part of the problem, we have:

$$\begin{aligned}
\angle W'WZ &= \arctan\left(\frac{2}{|(\sqrt{2}+1)-(-\sqrt{2}-1)|}\right) = \arctan\left(\frac{1}{\sqrt{2}+1}\right) = \arctan(\sqrt{2}-1) = \frac{\pi}{8}; \\
\angle W'WX &= \arctan\left(\frac{2}{|(-\sqrt{2}+1)-(-\sqrt{2}-1)|}\right) = \arctan 1 = \frac{\pi}{4}; \\
\angle X'XY &= \arctan\left(\frac{2}{|(\sqrt{2}-1)-(-\sqrt{2}+1)|}\right) = \arctan\left(\frac{1}{\sqrt{2}-1}\right) = \arctan(\sqrt{2}+1) = \frac{3\pi}{8}.
\end{aligned}$$

Therefore

$$\operatorname{area}(WXYZ) = 2\pi + 2\angle W'WZ - 4\angle W'WX - 2\angle X'XY = 2\pi + 2\cdot\frac{\pi}{8} - 4\cdot\frac{\pi}{4} - 2\cdot\frac{3\pi}{8} = \frac{\pi}{2}$$

as claimed.