Math 331, Section 201—Homework #3
due in class Tuesday, February 25, 2003

Warm-Up Questions—do not hand in

Graham, Knuth, and Patashnik, Chapter 7, pp. 371–375, #2, #3, #5, #7, #11, #26

Homework Questions—turn in solutions to these problems

I. Find a closed formula for the quantity

$$\sum_{k=0}^{n} (-1)^k \binom{r}{k} \binom{r}{n-k}$$

valid for all nonnegative integers n and all real numbers r.

II. Prove equation (5.24) on page 169 of Graham, Knuth, and Patashnik directly using generating functions.

III. Find a differential equation satisfied by the generating function of the sequence $\langle b_n \rangle$ that is defined by the recurrence

$$b_0 = 6, \quad b_1 = 23, \quad (n-1)b_n = 2nb_{n-1} - (n+2)(n+3) \quad (n \ge 2).$$

Then find a closed formula for b_n by showing that the generating function you found in problem I of Team Problem #3 satisfies this differential equation. (You may assume anything found in the solution set to Team Problem #3, as long as you understand what you are citing.)

IV. Find a closed formula for the quantity

$$\sum_{k \le n/2} (-1)^k \binom{n-k}{k} 3^{n-2k}$$

valid for all nonnegative integers n.

V. Suppose that a sequence $\langle s_n \rangle$ satisfies the recurrence

$$s_n = cs_{n-1} + p(n)$$

for some constant c and some polynomial $p(n)$ of degree $d \ge 0$. Prove that $\langle s_n \rangle$ also satisfies a recurrence of the form

$$s_n = c_1 s_{n-1} + c_2 s_{n-2} + \cdots + c_{d+2} s_{n-(d+2)}$$

for some constants $c_1, c_2, \dots, c_{d+2}$. (To avoid worrying about how many "base case" values need to be specified, you may assume that the first recurrence holds for n sufficiently large and prove that the second recurrence holds for n sufficiently large.)

VI. Bonus Problem: If $\langle f_n \rangle = \langle 0, 1, 1, 2, 3, 5, 8, 13, \dots \rangle$ denotes the Fibonacci sequence, use generating functions to find a simple formula for $f_k^2 + f_{k-1}^2$ for all positive integers k. (Hint: consider the more general expression $f_k f_{n-k} + f_{k-1} f_{n-k-1}$ for all positive integers n and $k \le n-1$, and use generating functions in two variables.)