Math 331, Section 201—Homework #4
due in class Tuesday, March 11, 2003

Warm-Up Questions—do not hand in

Graham, Knuth, and Patashnik, Chapter 6, pp. 309–313, #4, #11, #12, #23, #31, #32, #33. In #33, try to derive the closed form for $\left[{n \atop 2}\right]$ rather than $\left[{n \atop 3}\right]$.

Homework Questions—turn in solutions to these problems

I. Find a closed formula for $\left\{{n+2 \atop n}\right\}$. Find formulas for the generating function $G(z)$ and the exponential generating function $\hat{G}(z)$ of the sequence $\langle\left\{{n+2 \atop n}\right\}\rangle$.

II. Find a closed formula for the expression
$$\sum_j \left[{\ell \atop j}\right] \sum_k \left[{m \atop n-k}\right] \binom{j}{k} m^{j-k}$$
valid for nonnegative integers ℓ, m, n. (Hint: the presence of the term $n-k$ suggests moving the sum over k to the outside and writing the resulting sequence as a convolution. One of the resulting sequences will still be somewhat complicated, but it is given by a sum over j; what generating function method does that suggest?)

III. Given a nonnegative integer m, find a closed formula for the sequence $\langle h_n \rangle$ given by
$$h_n = \sum_k \binom{n}{k} \left[{m \atop m-k}\right] \binom{m-1}{k}^{-1} m! \left\{{n-k \atop m}\right\}.$$
You may use any of the formulas in Table 351 of Graham, Knuth, and Patashnik.

IV. Evaluate the constant
$$\sum_{n\ge1} \Big(\frac{(2\pi)^{2n}|B_{2n}|}{(2n)!} - 2 \Big).$$
You may use the formula for $\zeta(2n)$ from Team Problem #4. (Hint: rather than trying to use generating functions, substitute in the definition of $\zeta(2n)$ and exchange the order of summation. Always exchange the order of summation!—it's a philosophy of life.)

V. Given positive integers m and n, prove that
$$\sum_{0\le k<n} \frac{1-(k/n)^m}{1-k/n} = nH_m + \sum_{1\le k<m} \frac{B_k}{kn^{k-1}} \left(\binom{m}{k} - 1 \right).$$
(Hint: the summand on the left looks like the sum of a finite geometric series.)