Math 331, Section 201
Solutions for Team Problem #4
(March 6, 2003)

The Riemann zeta-function $\zeta(s)$ is defined by the formula

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

In this team problem you will recreate Euler's proof of the formula for $\zeta(2n)$, involving the Bernoulli numbers B_n. You may assume the following facts, the first of which you should know already but the second of which is somewhat more advanced:

$$\sin t = \frac{e^{it} - e^{-it}}{2i} \tag{$*$}$$

where $i = \sqrt{-1}$, and

$$\sin \pi x = \pi x \prod_{n=1}^{\infty} \Big(1 - \frac{x^2}{n^2}\Big). \tag{$**$}$$

Note: many of the manipulations in the following solutions are not valid for *arbitrary* infinite sums and products. Nevertheless, they are valid for *these* sums and products; since such details are not the main point of this exercise, we do not dwell on them below.

I. Use the formulas ($*$) and ($**$), with judicious choices of x and t, to prove the formula

$$e^{z/2} - e^{-z/2} = z \prod_{n=1}^{\infty} \Big(1 + \frac{z^2}{4\pi^2 n^2}\Big).$$

Setting $t = z/2i$ in formula ($*$) gives

$$\sin \frac{z}{2i} = \frac{e^{z/2} - e^{-z/2}}{2i},$$

while setting $x = z/2\pi i$ in the second formula gives

$$\sin \frac{z}{2i} = \frac{z}{2i} \prod_{n=1}^{\infty} \Big(1 - \frac{(z/2\pi i)^2}{n^2}\Big).$$

Comparing these two formulas gives

$$\frac{e^{z/2} - e^{-z/2}}{2i} = \frac{z}{2i} \prod_{n=1}^{\infty} \Big(1 + \frac{z^2}{4\pi^2 n^2}\Big),$$

and multiplying through by $2i$ gives the desired result.

II. Multiply the previous formula through by $e^{z/2}$, and then take logarithmic derivatives of both sides (this means first take the logarithm, then take the derivative of the result). In this way prove the formula

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{n=1}^{\infty} \frac{2z^2}{4\pi^2 n^2 + z^2}.$$

Multiplying through by $e^{z/2}$ yields

$$e^z - 1 = ze^{z/2} \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{4\pi^2 n^2}\right),$$

which becomes (upon taking logarithms)

$$\ln(e^z - 1) = \ln z + \frac{z}{2} + \sum_{n=1}^{\infty} \ln\left(1 + \frac{z^2}{4\pi^2 n^2}\right).$$

Taking derivatives term by term yields

$$\frac{e^z}{e^z - 1} = \frac{1}{z} + \frac{1}{2} + \sum_{n=1}^{\infty} \left(1 + \frac{z^2}{4\pi^2 n^2}\right)^{-1} \cdot \frac{2z}{4\pi^2 n^2} = \frac{1}{z} + \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2z}{4\pi^2 n^2 + z^2}.$$

If we subtract 1 from both sides, we get

$$\frac{e^z}{e^z - 1} - 1 = \frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2z}{4\pi^2 n^2 + z^2},$$

whereupon multiplying through by z results in the desired formula.

III. Considering both sides of the previous formula as generating functions, deduce that the only nonzero odd-numbered Bernoulli number is $B_1 = -\frac{1}{2}$. Deduce that the even-numbered Bernoulli numbers alternate in sign starting with B_2, that is, $B_2 > 0$, $B_4 < 0$, $B_6 > 0$, and so on. Finally, deduce the marvelous formula

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{2^{2k-1}\pi^{2k}|B_{2k}|}{(2k)!}$$

valid for all integers $k \geq 1$.

We've seen in class that

$$\frac{z}{e^z - 1} = \sum_{k=1}^{\infty} \frac{B_k}{k!} z^k, \tag{†}$$

so we can interpret the left-hand side as the generating function of the sequence $\langle B_k/k! \rangle$. As for the right-hand side, we use the geometric series expansion

$$\frac{2z^2}{4\pi^2n^2 + z^2} = \frac{z^2}{2\pi^2n^2}\left(1 - \frac{-z^2}{4\pi^2n^2}\right)^{-1} = \frac{2z^2}{4\pi^2n^2}\sum_{k=0}^{\infty}\left(\frac{-z^2}{4\pi^2n^2}\right)^k = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}z^{2k}}{2^{2k-1}\pi^{2k}n^{2k}}$$

to obtain

$$\begin{aligned}
1 - \frac{z}{2} + \sum_{n=1}^{\infty}\frac{2z^2}{4\pi^2n^2 + z^2} &= 1 - \frac{z}{2} + \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{(-1)^{k-1}z^{2k}}{2^{2k-1}\pi^{2k}n^{2k}}\right) \\
&= 1 - \frac{z}{2} + \sum_{k=1}^{\infty}\frac{(-1)^{k-1}z^{2k}}{2^{2k-1}\pi^{2k}}\left(\sum_{n=1}^{\infty}\frac{1}{n^{2k}}\right) \\
&= 1 - \frac{z}{2} + \sum_{k=1}^{\infty}\frac{(-1)^{k-1}\zeta(2k)z^{2k}}{2^{2k-1}\pi^{2k}}. \qquad (††)
\end{aligned}$$

Comparing the coefficients of the two formal power series (†) and (††), we immediately conclude that $B_0 = 1 \cdot 0! = 1$ and $B_1 = -\frac{1}{2} \cdot 1! = -\frac{1}{2}$, and that $B_{2k+1} = 0$ for all $k \geq 1$. Finally, we have

$$\frac{B_{2k}}{(2k)!} = \frac{(-1)^{k-1}\zeta(2k)}{2^{2k-1}\pi^{2k}}$$

for all $k \geq 1$, and solving for $\zeta(2k)$ yields

$$\zeta(2k) = \frac{2^{2k-1}\pi^{2k}(-1)^{k-1}B_{2k}}{(2k)!}.$$

Now $\zeta(2k)$ is clearly positive from its definition, and so we conclude that $(-1)^{k-1}B_{2k}$ is also positive, that is, B_{2k} has the same sign as $(-1)^{k-1}$ and $(-1)^{k-1}B_{2k} = |B_{2k}|$. This establishes the last assertions of the problem.