Math 331–Homework #4

due at the beginning of class Monday, March 5, 2007

Before you do anything else: go to Wilf, p. 16, and memorize equation (1.5.5)! Remember that the first identity of (1.5.5) holds for any real number n, while the second one holds only for nonnegative integers k. Also scan briefly through Wilf, pp. 52–53.

Reality-check problems. Not to write up; just ensure that you know how to do them.

I. Verify that $\frac{1}{2j-1}\binom{2j}{j} = \frac{2}{j}\binom{2(j-1)}{j-1}$ for any nonzero integer j.

II. (a) Wilf, p. 105, #5(a) and (c)
 (b) Show that Wilf, p. 105, #5(a) follows directly from Theorem 3.10.1.

III. Wilf, pp. 105–106, #10 and #12

IV. Show that the following are all equal: the number of partitions of n where all the even summands are distinct; the number of partitions of n where no summand occurs more than three times; and the number of partitions of n into summands that aren't multiples of 4.

V. Wilf, pp. 25–26, #10(a)–(d)

VI. Wilf, p. 160, #11(f)

Homework problems. To write up and hand in.

I. Let $\beta(n)$ denote the number of vertex-labeled bipartite graphs with n vertices. Prove that for every constant $C > 0$ no matter how large, there are infinitely many integers n such that $\beta(n) > n! \cdot C^n$. (Hint: consider the radius of convergence of the exponential generating function of $\{\beta(n)\}$. How is this related to the radius of convergence of the exponential generating function of $\{\gamma(n)\}$, where $\gamma(n)$ is the number of vertex-labeled, vertex-colored bipartite graphs with n vertices?)

II. For each positive integer m, let $e_m(n)$ be the number of permutations π of $\{1, \dots, n\}$ whose order is exactly m (that is, π^m is the identity permutation, but no smaller power π^k is), and let $E_m(x)$ be the exponential generating function of $e_m(n)$.
 (a) Prove that $E_8(x) = e^{x+x^2/2+x^4/4}\big(e^{x^8/8} - 1\big)$.
 (b) What is $E_6(x)$? What is $E_1(x)$?

III. Find the exponential generating functions of
 (a) $f_{\text{odd}}(n)$, the number of permutations of $\{1, \dots, n\}$ with an odd number of cycles in their disjoint cycle decomposition; and
 (b) $f_{\text{even}}(n)$, the number of permutations of $\{1, \dots, n\}$ with an even number of cycles in their disjoint cycle decomposition.

From these exponential generating functions, conclude that $f_{\text{odd}}(n) = f_{\text{even}}(n)$ for all $n \geq 2$.

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IV. A particular magnetic poetry kit consists of n words, all different from one another. At a party with k people in attendance, the partygoers divide all the magnetic poetry words up among themselves (each partygoer gets at least one word). The partygoers then arrange the words they got into a "poem", and the poems are left on the side of the refrigerator anonymously. Prove, by interpreting this scenario as an exponential family, that there are exactly

$$\frac{n!(n-1)!}{k!(n-k)!(k-1)!}$$

different sets of k poems that can be produced in this way.

To clarify, the order of the words in each poem matters, but the order of the poems themselves doesn't matter—they just all end up on the fridge together. And we're assuming that any word ordering results in a poem, not just sensible word orderings. Finally, there is a combinatorial proof of this formula, but you should specifically use the cards-decks-hands method from Chapter 3.

V. Recall that every permutation π induces a partition of $\{1,\ldots,n\}$, simply by replacing a cycle of π with the corresponding subset of $\{1,\ldots,n\}$.

(a) Let $s(n)$ denote the number of permutations π of $\{1,\ldots,n\}$ such that π and $\pi^2 = \pi \circ \pi$ induce the same partition of $\{1,\ldots,n\}$. (Define $s(0) = 1$.) Show that the exponential generating function of $\{s(n)\}$ is

$$\sqrt{\frac{1+x}{1-x}}.$$

(b) Using the Snake Oil Method, prove that

$$\sum_j \frac{(-1)^j}{1-2j}\binom{2j}{j}\binom{2(n-j)}{n-j} = \frac{4^n s(n)}{n!}.$$

VI. Find a formula in closed form for $\sum_k \binom{n-k}{k} 2^k$.

VII. Wilf, p. 160, #11(d) and (e)

VIII. Wilf, pp. 160–161, #12(a) and (c)

IX. Let $p(n)$ be the number of partitions of the integer n into (a sum of) positive integers, with $P(x) = \sum_{n=0}^{\infty} p(n)x^n$ the corresponding ordinary generating function, and let $p(n,k)$ be the number of partitions of n into k positive integers. Let

$$t(n) = \sum_k k\,p(n,k)$$

denote the total number of summands of all partitions of n, and define the ordinary generating function $T(x) = \sum_{n=0}^{\infty} t(n)x^n$. Show that

$$T(x) = P(x) \sum_{n=1}^{\infty} d(n)x^n,$$

where $d(n)$ is the number of positive divisors of n.