Math 331–Homework #5

due at the beginning of class Monday, March 19, 2007

Reality-check problems. Not to write up; just ensure that you know how to do them.

I. Verify that $\exp_{|\alpha}(0)$ equals 1 if $\alpha \geq 0$ and 0 if $\alpha < 0$.

II. Verify that for any function $f(x)$,

$$\frac{d}{dx}\big(\exp_{|\alpha}(f(x))\big) = f'(x)\exp_{|\alpha-1}(f(x)).$$

III. Wilf, p. 158, #4

IV. Wilf, p. 159, #10. Note: this isn't an easy problem, but it's an important example. I've just called it a Reality Check problem to indicate that you don't have to write it up and turn it in. But I suggest you do work through it!

Homework problems. To write up and hand in.

I. Wilf, p. 161, #14

II. Wilf, p. 158, #7. Note that the sieve formula for e_k mentioned in the problem is equation (4.2.7).

III. (a) Let N_r and E_t have their usual meanings from Section 4.2 of Wilf. We've seen that the average number of properties an object has is given by the simple formula N_1/N_0. Show that the variance of the number of properties is given by $(N_1 + 2N_2)/N_0 - (N_1/N_0)^2$.
(b) Using part (a), re-derive the formula for the variance of the number of k-cycles in permutations of $\{1, \ldots, n\}$.

IV. In a given sieve-method problem, the set of properties under consideration is a finite set P. Suppose we are given a positive number A, as well as other positive numbers c_p for every $p \in P$. Suppose also that the number $N(\supseteq S)$ of objects possessing all the properties in S is given by the formula $N(\supseteq S) = A\prod_{p\in S} c_p$. Prove that the "exactly" generating function $E(x)$ is given by

$$E(x) = A\prod_{p\in P}\big(c_p x + 1 - c_p\big).$$

V. Let $p_1, \ldots, p_m$ be distinct prime numbers, and let B be a positive integer that is a multiple of all m of them. For any positive integer n, let $w(n)$ be the number of $p_1, \ldots, p_m$ that divide n. By phrasing this as a sieve method problem, answer the following:

(a) What is the average value of $w(n)$ as n ranges over $\{1, \ldots, B\}$?
(b) What is the variance of $w(n)$ as n ranges over $\{1, \ldots, B\}$?
(c) Using $E(x)$, find a formula for the number of n in the range $\{1, \ldots, B\}$ that are not divisible by any of the primes $p_1, \ldots, p_m$.

VI. Wilf, p. 158, #5

(continued on next page)

VII. Let h and w be positive integers (think "height" and "width"). Use generating functions, in the manner of Example 4 of Section 4.2 of Wilf, to prove:

(a) For every positive integer j, the number of ways to place j non-attacking rooks on an $h \times w$ chessboard is

$$\binom{h}{j}\binom{w}{j} j!.$$

(Hint: consider the $h \times w$ chessboard as a subset of a square chessboard of side length $\max\{h, w\}$. In this case, the "exactly" numbers are extremely easy to calculate!)

(b) For every integer $n \geq \max\{h, w\}$, the probability that a randomly chosen permutation π of $\{1, \ldots, n\}$ satisfies

$$\pi(1) > h, \ \pi(2) > h, \ \ldots, \text{ and } \pi(w) > h$$

is given by the formula

$$\sum_{k=0}^{n} (-1)^k \binom{h}{k}\binom{w}{k} \Big/ \binom{n}{k}.$$

VIII. In this problem (unlike the previous one), permutation probabilities refer to the limiting probability as n goes to infinity, as in Section 4.7. Show that:

(a) the probability that a permutation has no 2-cycles or 6-cycles is the same as the probability that a permutation has no 3-cycles, 4-cycles, or 12-cycles;

(b) the probability that a permutation's shortest cycle is the only cycle of that size in the permutation is

$$\sum_{n=1}^{\infty} \frac{1}{ne^{H_n}},$$

where $H_n = \sum_{k=1}^{n} 1/k$ is the nth harmonic number;

(c) for every positive integer k, it is more probable that a permutation has exactly k $(k+1)$-cycles than it is that a permutation has exactly $(k+1)$ k-cycles.