Math 432/537 Take-Home Final Exam
due Monday, December 16, 2002 by 3 pm

The due date to hand in your written solutions is Monday, December 16 by 3 pm. You can bring your written solutions to me in my office (or slip them under my door if I'm not there), or drop them off in the Math Office and ask them to put them in my mailbox. You are allowed to consult your notes, the textbook, etc. when working on the exam. However, you are not allowed to discuss the material from the course or the exam with your classmates or anybody else during the 10-day period specified above. Please take this restriction seriously.

Read the statements of the problems carefully to ensure that you are answering the question asked. Whenever you see parts (a) and (b) of a problem, you may always assume the result to be proved in part (a) when you are solving part (b), even if you did not completely solve part (a). The parts are intended to be related to each other, and so earlier parts often can be interpreted as hints or important special cases of later parts, although you are free to seek unrelated solutions as well. An encouraging word: every time you start thinking about one of these problems, remind yourself that I've chosen the problem specifically for you to be able to solve using knowledge from this course.

You may use any results proved in the textbook or on your homework assignments—simply cite the general fact you are using (clearly enough for me to be convinced that you know what happens in general). For any computational questions, the problem will make it clear how much computational detail you have to show. You are not allowed to use commands like "solve" or "contfrac" in Maple or Mathematica or to do extensive brute-force searches, etc. Using a calculator is fine.

I will be out of town from December 7–11. However, I will be checking my email during that time. If there is a problem with the exam for any reason, please email me. I will send email to all of you if any comments or corrections need to be made while I am away, and I will also post them prominently on the course web page.

1. Find all positive integer solutions to the equation $336x + 483y = 46179$. (Show all your work.)

2. Let d and n be positive integers with n odd and $d \mid n$. Prove that there exist integers a, b, both relatively prime to n, such that $(a + b, n) = d$. (Hint: Chinese Remainder Theorem.) *[Editor's note: in the first version of this final exam, the condition that n be odd was mistakenly omitted.]*

3. Let $c = 2061^{(1494^{255})}$. When c is written in base 7, what are its last three digits? (Show the results of all your calculations, but you may use a calculator to do arithmetic.)

4. Do not do these problems by examining all possible residue classes individually.
   (a) How many solutions are there to the congruence $x^{70} \equiv 609 \pmod{1952}$? (Note that $1952 = 32 \times 61$.)
   (b) How many 15th powers are there modulo 144875? In other words, for how many residue classes $x \pmod{144875}$ does there exist an integer y such that

$y^{15} \equiv x \pmod{144875}$? (Note that $144875 = 19 \times 61 \times 125$.) *[Editor's note: the first version of this final had a typo in this question; it mistakenly referred to a modulus 28975 instead of 144875.]*

5. Let $q(x) = x^5 + 3x^3 - 2x^2 - 6 = (x^2 + 3)(x^3 - 2)$.
   (a) Prove that $q(x) \equiv 0 \pmod{p}$ has a solution for every prime p.
   (b) Suppose m is an integer that is congruent to 1 or 5 modulo 6. Prove that $q(x) \equiv 0 \pmod{m}$ has a solution.

6. (a) Let $f(x, y) = 558x^2 + 2495xy + 2789y^2$. Find a reduced binary quadratic form that is equivalent to f. Calculate the discriminant of f. Show all your work.
   (b) Find a proper representation of the prime 2789 by the form $2x^2 + xy + 3y^2$. (Do not use trial and error.)

7. Define $h(n) = d(n)^2$, where $d(n)$ denotes the number of positive divisors of n. Write down a *simple* definition of an arithmetic function $g(n)$ such that $h(n) = \sum_{d|n} g(d)$ for all positive integers n. Evaluate $g(11!)$. (The point of this last evaluation is to make sure that your definition of $g(n)$ is simple enough to work with. For example, it should not be necessary to find all of the divisors of $11!$.)

8. (a) Find all pairs of integers x, y satisfying the equation $y^2 = x(x+1)(x+2)(x+4)$. (Note that $x(x+1)(x+2)(x+4) = x^4 + 7x^3 + 14x^2 + 8x$. There are six solutions in all.)
   (b) Suppose that you start with five consecutive positive integers, choose four of them to multiply together, and end up with a perfect square. Prove that the four chosen integers are $\{2, 3, 4, 6\}$. (Hint: there is more than one case, depending on which of the five integers is left out. Some of these cases may possibly be combined.)

9. (a) Suppose that w, x, y, z are integers such that $w^4 + x^4 + y^4 + z^4 \equiv 0 \pmod{5}$. Prove that $w \equiv x \equiv y \equiv z \equiv 0 \pmod{5}$.
   (b) Prove *carefully* that the only integer solution of the equation $w^4 + x^4 + y^4 + z^4 = 35wxyz$ is $w = x = y = z = 0$. *[Editor's note: the first version of this final had a typo in this question; it mistakenly referred to the equation $w^4+x^4+w^4+z^4 = 35wxyz$.]*

10. Find three solutions in positive integers to the equation $x^2 - 77y^2 = -7$. (Show your work when calculating any continued fractions or convergents, but you may use a calculator to do arithmetic.)

11. (a) Let $\alpha = 1 + \sqrt{3}$. Find the largest constant C with the following property: there are infinitely many rational numbers $\frac{a}{b}$ such that $\left|\alpha - \frac{a}{b}\right| < \frac{1}{Cb^2}$. Justify your answer. (Show your work when calculating any continued fractions or convergents, but you may use a calculator to do arithmetic.)

(b) Let $\beta = \frac{1+\sqrt{5}}{2}$ be the golden ratio. Find the smallest constant C with the following property: there are no rational numbers $\frac{a}{b}$ with $|\beta - \frac{a}{b}| < \frac{1}{Cb^2}$. Justify your answer.