Math 432/537 Homework #4

due Friday, November 1, 2002 at the beginning of class

Remark: in this homework assignment, "number" means "positive integer". ("Integer" still means "positive, negative, or zero integer".)

I. Niven, Zuckerman, and Montgomery, Section 3.5, p. 162, #8

II. Compute the class number $H(-71)$. Find a reduced binary quadratic form of discriminant -71 that is equivalent to the form $116x^2 + 141xy + 43y^2$.

III. (a) Find the smallest number n with exactly 42 positive divisors.

(b) Find the smallest number n such that there are exactly 48 ordered pairs (a, b) of integers with $a^2 + b^2 = n$.

(c) Find all numbers n such that $\phi(n) = 120$.

IV. Prove that $\phi(n) + d(n) \le n + 1$ for all integers n. (Hint: think about the definitions of $\phi(n)$ and $d(n)$.)

V. Prove that for every number n, there is a number x such that $d(nx) = n$.

VI. One can classify all numbers into three types: deficient, perfect, and abundant. A number n is deficient, perfect, or abundant depending on whether the sum of the positive divisors of n (other than n itself) is less than n, equal to n, or greater than n, respectively. Prove that there are infinitely many deficient numbers and infinitely many abundant numbers. (If you can prove that there are infinitely many perfect numbers, you can be a math professor anywhere you want!)

VII. If $a^k - 1$ is prime, show that $a = 2$ and that k is prime. If $a^k + 1$ is prime, show that a is even and that k is a power of 2. (Hint: consider factorizations of the polynomials $x^k \pm 1$.)

VIII. (a) Suppose that $n \equiv 5 \pmod 6$. Prove that if d is any divisor of n, then $d + n/d$ is a multiple of 3. Conclude that $\sigma(n) \not\equiv 2n \pmod 3$.

(b) Suppose that $n \equiv 3 \pmod 4$. Prove that $\sigma(n) \not\equiv 2n \pmod 4$.

(c) Suppose that n is an odd perfect number. Prove that either $n \equiv 1 \pmod{12}$ or $n \equiv 9 \pmod{36}$. (Hints: use parts (a) and (b) to determine what n can be congruent to modulo 12. If $n = 12m + 9$ and $3 \nmid m$, prove that $\sigma(n) \equiv 0 \not\equiv 2n \pmod 4$.)

(d) Bonus: if n is an odd perfect number, prove that n can be written as $n = p^r m^2$ where $p \equiv r \equiv 1 \pmod 4$ and $p \nmid m$.

IX. Define $f(n) = \sum_{d|n} \phi\big((d, n/d)\big)$ for all numbers n. (That's the Euler phi-function applied to a gcd.)

(a) Prove that f is a multiplicative function.

(b) Prove that there exist multiplicative functions g and h such that $f(n) = \sum_{d|n} g(d)$ and $g(n) = \sum_{d|n} h(d)$. Furthermore, prove that for this function h, we have $h(n) \neq 0$ if and only if n is a perfect square.