Math 437/537 Homework #5

due Friday, November 14, 2003 at the beginning of class

In this homework, there will be many occasions to use basic analytic tools such as the triangle inequality, comparing a sum to an integral, and other means of establishing simple inequalities. Don't be afraid to get your hands dirty.

I. Niven, Zuckerman, and Montgomery, Section 6.3, p. 311, #3

II. Niven, Zuckerman, and Montgomery, Section 6.4, p. 321, #18

III. (a) Let N be a positive integer. Prove that the number of ordered pairs of integers $1 \le m, n \le N$ that are relatively prime to each other is exactly

$$\sum_{d \le N} \mu(d) \left\lfloor \frac{N}{d} \right\rfloor^2.$$

(Hint: consider the sum $\sum_{d|(m,n)} \mu(d)$.)

(b) For any positive real number x and any positive integer N, prove that

$$\left| x^2 - \lfloor x \rfloor^2 \right| \le 3x \quad \text{and} \quad \left| \sum_{d > N} \frac{\mu(d)}{d^2} \right| < \frac{1}{N}.$$

(The two assertions are to be proved separately; they are not directly related.)

(c) Define

$$S(N) = \frac{1}{N^2} \sum_{d \le N} \mu(d) \left\lfloor \frac{N}{d} \right\rfloor^2 \quad \text{and} \quad Z = \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2}.$$

Prove that $\lim_{N\to\infty} S(N) = Z$. (Hint: establish some inequality like

$$|S(N) - Z| \le \frac{2}{N} + \frac{3 \log N}{N},$$

perhaps by comparing both $S(N)$ and Z to $Z(N) = \sum_{d=1}^{N} \mu(d)/d^2$.) Remark: it is known that $Z = 6/\pi^2$. This result can be interpreted as saying that "the probability of two randomly chosen integers being relatively prime to each other is $6/\pi^2$."

IV. Recall the theorem we proved in class, that for every real number x and every positive integer Q there exists a rational number a/q such that $1 \le q \le Q$ and $|x - a/q| \le 1/q(Q+1)$. Prove this theorem using the "geometry of numbers" technique. (Hint: consider the region

$$\left\{ (s,t) \colon |s| < Q+1, \ |sx - t| < \frac{1+\varepsilon}{Q+1} \right\}$$

in $\mathbb{R}^2$.)

(continued on back of page)

V. Define a sequence of integers $d_2, d_3, \dots$ by $d_2 = 2$ and

$$d_j = j^{\phi(d_{j-1})} \quad (j \geq 3).$$

Using this sequence, define a sequence of rational numbers $\alpha_2, \alpha_3, \dots$ by

$$\alpha_k = \prod_{j=2}^{k} \left(1 - \frac{1}{d_j}\right).$$

For instance, $\{d_2, d_3, \dots\} = \{2, 3, 16, 5^8, \dots\}$ and $\{\alpha_2, \alpha_3, \dots\} = \{\frac{1}{2}, \frac{1}{3}, \frac{5}{16}, \dots\}$.

(a) Prove that for all $k \geq 2$, the number α_k is a rational number with denominator d_k (not necessarily in lowest terms).

(b) Prove that for every $\ell > k \geq 2$, we have

$$\alpha_k > \alpha_\ell > \alpha_k \left(1 - \frac{2}{d_{k+1}}\right).$$

(Hint: you could try proving the inequality

$$\prod_{i=1}^{m} (1 - x_i) \geq 1 - \sum_{i=1}^{m} x_i,$$

which is valid for $0 \leq x_1, x_2, \dots, x_m \leq 1$. The crude inequality $d_{j+1} > 2d_j$ might also be useful to prove.)

(c) Conclude that the limit

$$\alpha = \lim_{k \to \infty} \alpha_k = \prod_{j=2}^{\infty} \left(1 - \frac{1}{d_j}\right)$$

exists and is irrational. (Hint: each α_k is ridiculously close to α....)