Math 437/537 Homework #6
due Wednesday, December 3, 2003 at 3 PM

In this homework, I want to see all your calculations done by hand. (You may use a calculator to do the arithmetic and just record the answers; but show all the steps, and I don't want any computer searches, programmed calculations, and so on.)

I. Niven, Zuckerman, and Montgomery, Section 7.3, p. 333, #4 and #6

II. Niven, Zuckerman, and Montgomery, Section 7.5, p. 341, #6

III. Of the first 1,000 positive multiples of e, which one is closest to an integer?

IV. Compute the real number whose continued fraction expansion is $\langle 4, 3, \overline{2, 1, 2} \rangle$.

V. In this problem, "smallest solution" means the ordered pair (x, y) of positive integers with y as small as possible.
   (a) Compute the (somewhat complicated) continued fraction expansion of $\sqrt{61}$.
   (b) Find the smallest solution to $x^2 - 61y^2 = -1$.
   (c) Find the smallest solution to $x^2 - 61y^2 = 1$. (Hint: derive it directly from your answer to part (b).)

VI. Given an integer $k \geq 2$, compute the continued fraction expansions of $\sqrt{k^2 - 1}$ and $\sqrt{k^2 + 4}$.

VII. Prove that the equation $x^2 + 1 = (k^2 + 4)y^2$ has no integer solution (x, y) if k is even, but infinitely many integer solutions if k is odd.

VIII. Given integers $1 \leq m \leq n$, define $x_{m,n} = \langle 0, 1, \ldots, m - 1, \overline{m, \ldots, n} \rangle$.
   (a) Put the infinite set of real numbers $\{x_{m,n} : 1 \leq m \leq n\}$ in order by size.
   (b) Prove that if $n \leq s$, then $|x_{m,n} - x_{r,s}| < 2/(n!)^2$ for all $1 \leq m \leq n$ and $1 \leq r \leq s$.

(continued on back of page)

IX. In Triangle Town, the streets are named 2nd Avenue, 3rd Avenue, 4th Avenue, and so on. There are exactly n houses on nth Avenue, the numbers of which (from west to east) are $1, 2, \ldots, n$. Alice, Bob, and Eve have the following conversation:

> BOB: What street do you live on, Alice?
>
> *(Alice whispers the answer to Bob so that Eve does not hear)*
>
> BOB: What's the number of your house, Alice?
>
> ALICE: I won't tell you the number, but I will tell you how to find my house. Start at one end of my street and walk down it, adding up the numbers of all the houses one by one as you come to them. When your running total equals the following number . . . *(Alice whispers a number to Bob so that Eve does not hear)* . . . then my house is the last house you added in. *(Alice starts to walk away)*
>
> BOB: Wait, Alice: which end of your street do I start from, the west end or the east end?
>
> ALICE: That's the best part: it doesn't matter! *(Alice leaves)*
>
> EVE: I know you don't want to tell me Alice's address, Bob, but at least tell me this much: how many digits are there in her street number?
>
> *(Bob answers Eve's question)*

What is Alice's address (street and house number) if Bob's answer was "One digit"? if Bob's answer was "Two digits"? if Bob's answer was "Three digits"? (Hint: complete the square.)