Math 437/537 Homework #2

due Friday, September 28, 2007 at the beginning of class

For all of these problems, show all of your calculations; do not use brute-force or exhaustive approaches, and do not use a computer (although using a calculator for arithmetic is fine).

I. (a) Find all solutions to each of the following congruences (individually):

$76x \equiv 90 \pmod{105}$; $77x \equiv 91 \pmod{105}$; $78x \equiv 92 \pmod{105}$.

(b) Find all lattice points on the line $77x - 105y = 91$. Which of these lattice points have the property that the line segment connecting them to the origin contains other lattice points?

II. A *squarefree number* is an integer that is not divisible by any nontrivial square; that is, $n \in \mathbb{Z}$ is squarefree if and only if $d^2 \mid n$ implies $d = \pm 1$. Prove that there are arbitrarily large gaps between consecutive squarefree numbers. (Hint: Chinese Remainder Theorem.)

III. Niven, Zuckerman, and Montgomery, Section 2.6, p. 91, #7. (Use Hensel's Lemma; show your work.)

IV. Prove that every integer of the form $x^{18} + y^{18}$ lies in one of 15 residue classes modulo 703.

V. (a) If p is a prime, how many solutions are there to the congruence $x^4 - x^3 + x^2 - x + 1 \equiv 0 \pmod{p}$? The answer should only depend on the last digit of p. (Hint: factor the polynomial $x^{10} - 1$.)

(b) How many solutions are there to the congruence

$$x^4 - x^3 + x^2 - x + 1 \equiv 0 \text{ (mod 2,269,355)?}$$

VI. Let a, b, and m be integers with $m \neq 0$ and $(a, m) = (b, m) = 1$. Let r denote the order of $a \pmod{m}$, let s denote the order of $b \pmod{m}$, and let t denote the order of $ab \pmod{m}$. Prove that

$$\frac{rs}{(r,s)^2} \mid t \quad \text{and} \quad t \mid \frac{rs}{(r,s)}.$$

VII. Niven, Zuckerman, and Montgomery, Section 2.8, p. 107, #24

VIII. Niven, Zuckerman, and Montgomery, Section 2.8, p. 107, #26

IX. Find all Carmichael numbers of the form $3pq$ where p and q are prime.

X. Niven, Zuckerman, and Montgomery, Section 2.8, p. 109, #37

XI. Show that the only consecutive powers of 2 and 3 are $(1, 2)$, $(2, 3)$, $(3, 4)$, and $(8, 9)$. (Hint: consider the equations $2^m - 3^n = \pm 1$ and the order of 2 modulo 3^n.)

XII. Determine the number of solutions to the congruence $22x^2 + 10x + 35 \equiv 0 \pmod{p}$ for each of the primes $p = 103$, $p = 149$, and $p = 227$.