Math 437/537 Homework #4

due Monday, October 29, 2007 at the beginning of class

I. Prove the following theorem of Dickson from 1894: (x, y, z) is a primitive Pythagorean triple if and only if $(x, y, z) = (\rho + \tau, \sigma + \tau, \rho + \sigma + \tau)$ for some positive integers ρ, σ, τ satisfying $(\rho, \sigma) = 1$ and $\tau^2 = 2\rho\sigma$. Given such a triple $(\rho + \tau, \sigma + \tau, \rho + \sigma + \tau)$, how can we recover the values of r and s that give the representation in Theorem 5.5?

II. (a) Niven, Zuckerman, and Montgomery, Section 5.3, p. 233, #10
(b) Niven, Zuckerman, and Montgomery, Section 5.4, p. 240, #10

III. Niven, Zuckerman, and Montgomery, Section 5.4, p. 239, #6 and #7 (the two problems go together naturally)

IV. Let $g(x, y) = x + 2x^2 + 3x^3 + 3y + y^3$.
(a) Two points such that $g(x, y) = 10$ are $(1, 1)$ and $(-3, 4)$. Find a third point (u, v) with rational coordinates such that $g(u, v) = 10$ by considering the line passing through the first two points.
(b) Find two points (u, v) with rational coordinates such that $g(u, v) = 0$. One such point should be obvious; find the second by considering the tangent line to the curve $g(x, y) = 0$ passing through the first point.

V. Niven, Zuckerman, and Montgomery, Section 5.6, p. 260, #5. In addition, find two nonzero rational numbers x, y such that $y^2 = x^3 + 2x^2$ and both $|x|$ and $|y|$ are less than $\frac{1}{100}$.

VI. (a) Find the smallest number n such that there are exactly 48 ordered pairs (a, b) of integers with $a^2 + b^2 = n$.
(b) Find all numbers n such that $\phi(n) = 120$.

VII. Prove that for every number n, there is a number x such that $\tau(nx) = n$.

VIII. Niven, Zuckerman, and Montgomery, Section 4.3, p. 196, #18

IX. Define $f(n) = \sum_{d|n} \phi\big((d, n/d)\big)$ for all numbers n. (That's the Euler phi-function applied to a gcd.)
(a) Prove that f is a multiplicative function.
(b) Prove that there exist multiplicative functions g and h such that $f(n) = \sum_{d|n} g(d)$ and $g(n) = \sum_{d|n} h(d)$. Furthermore, prove that for this function h, we have $h(n) \neq 0$ if and only if n is a perfect square such that $2^2 \nmid n$.

X. The largest perfect number known today is $n_{44} = 2^{p-1}(2^p - 1)$, where $p =$ 32,582,657. Determine how many digits n_{44} has, and find the first three digits (on the left) and the last three digits (on the right). You may use a calculator to do arithmetic; just indicate what calculations you did. Do not evaluate n_{44} directly.

(continued on back of page)

XI. For any positive integer n, define $r_1(n)$ to be the number of positive divisors of n that are congruent to 1 (mod 3) and $r_2(n)$ to be the number of positive divisors of n that are congruent to 2 (mod 3). Find, with proof, the smallest positive integer n that is relatively prime to 6 and satisfies $r_1(n) > r_2(n) > 0$.

XII. When n is a positive integer, a *primitive nth root of unity* is a complex number of the form $e^{2\pi ik/n}$ where $(k,n) = 1$. Equivalently, a primitive nth root of unity is a complex number z such that $z^n = 1$ but $z^m \neq 1$ for any $0 < m < n$. Define the nth cyclotomic polynomial $\Phi_n(x)$ to be the polynomial whose roots are precisely the $\phi(n)$ primitive nth roots of unity. For example, $\Phi_6(x) = (x - e^{\pi i/3})(x - e^{5\pi i/3}) = x^2 - x + 1$.

(a) Prove that

$$\Phi_n(x) = \prod_{d|n} (x^d - 1)^{\mu(n/d)}.$$

For example, $\Phi_6(x) = (x-1)^1(x^2-1)^{-1}(x^3-1)^{-1}(x^6-1)^1 = x^2 - x + 1$.

(b) Define the *von Mangoldt Lambda-function*

$$\Lambda(n) = \begin{cases} \ln p, & \text{if } n = p^r \text{ with } p \text{ prime and } r \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

So for example, $\Lambda(125) = \ln 5$. Prove the two identites

$$\sum_{d|n} \mu(n/d) \ln d = \Lambda(n) \quad \text{and} \quad \sum_{d|n} \mu(d) \ln d = -\Lambda(n).$$

(Hint: $\ln d = \ln n - \ln(n/d)$.)

(c) Evaluate $\Phi_n(1)$ for every $n \geq 1$. (Hint: cancel factors of $x - 1$.)