Math 437/537 Homework #5

due Monday, November 10, 2008 at the beginning of class

I. (a) Prove that $\sum_{j=1}^{\infty} 2^{-3^j}$ is irrational.
   (b) Niven, Zuckerman, and Montgomery, Section 6.3, p. 312, #10

II. Define a sequence of integers $d_2, d_3, \dots$ by $d_2 = 2$ and

$$d_j = j^{\phi(d_{j-1})} \quad (j \ge 3).$$

Using this sequence, define a sequence of rational numbers $\alpha_2, \alpha_3, \dots$ by

$$\alpha_k = \prod_{j=2}^{k} \Big(1 - \frac{1}{d_j}\Big).$$

For instance, $\{d_2, d_3, \dots\} = \{2, 3, 16, 5^8, \dots\}$ and $\{\alpha_2, \alpha_3, \dots\} = \{\frac{1}{2}, \frac{1}{3}, \frac{5}{16}, \dots\}$.

(a) Prove that $d_k \alpha_k$ is an integer for all $k \ge 2$.

(b) Prove that for every $\ell > k \ge 2$, we have

$$\alpha_k > \alpha_\ell > \alpha_k \Big(1 - \frac{2}{d_{k+1}}\Big).$$

(Hint: you could try proving the inequality

$$\prod_{i=1}^{m} (1 - x_i) \ge 1 - \sum_{i=1}^{m} x_i,$$

which is valid for $0 \le x_1, x_2, \dots, x_m \le 1$. The crude inequality $d_{j+1} > 2d_j$ might also be useful to prove.)

(c) Conclude that the limit

$$\alpha = \lim_{k \to \infty} \alpha_k = \prod_{j=2}^{\infty} \Big(1 - \frac{1}{d_j}\Big)$$

exists and is transcendental. (Hint: each α_k is ridiculously close to α. ...)

III. Niven, Zuckerman, and Montgomery, Section 6.4, p. 320, #11

IV. Using the generalized Minkowski's convex body theorem, prove the following statement: given $(a, p) = 1$, there exist nonzero integers x and y with $|x| < \sqrt{p}$ and $|y| < \sqrt{p}$ such that $ax \equiv y \pmod{p}$. Hint: consider the lattice with corresponding matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & -1 & p \end{pmatrix}.$$

V. Niven, Zuckerman, and Montgomery, Section 6.1, p. 301, #9

VI. Niven, Zuckerman, and Montgomery, Section 7.2, p. 329, #1

(continued on next page)

VII. (a) Calculate the 0th, 1st, 2nd, 3rd, and 4th convergents to π, showing your work.

(b) Determine, with proof, the first 1,000 partial quotients in the continued fraction representation of $2\sqrt{2}$.

VIII. Niven, Zuckerman, and Montgomery, Section 7.3, p. 333, #4

IX. (a) Niven, Zuckerman, and Montgomery, Section 7.3, p. 333, #6

(b) Using the method in part (a) and the fact that $17682^2 \equiv -1 \pmod{100049}$, find a representation of the prime 100049 as the sum of two squares, showing your work.

X. Niven, Zuckerman, and Montgomery, Section 7.4, p. 336, #6

XI. For all nonnegative integers n, define

$$\psi_n(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{1 \cdot 3 \cdots (2k+2n-1) \cdot 2 \cdot 4 \cdots (2k)} \quad \text{and} \quad w_n(x) = \frac{\psi_n(x)}{x\psi_{n+1}(x)}.$$

Also define $u = w_0(1/2)$ and $v = (u-2)/(3-u)$.

(a) Show that $\psi_0(x) = (e^x + e^{-x})/2$ and $\psi_1(x) = (e^x - e^{-x})/(2x)$. Conclude that $w_0(x) = (e^x + e^{-x})/(e^x - e^{-x})$.

(b) Show that $e = (u+1)/(u-1) = \langle 2; 1+2v \rangle$.

(c) Show that $\psi_n(x) = (2n+1)\psi_{n+1}(x) + x^2\psi_{n+2}(x)$ for every $n \geq 0$. Conclude that $w_n(x) = (2n+1)/x + 1/w_{n+1}(x)$ for every $n \geq 0$.

(d) For every positive integer k, prove that

$$\frac{e^{1/k} + e^{-1/k}}{e^{1/k} - e^{-1/k}} = \langle k; 3k, 5k, 7k, 9k, \dots \rangle.$$

Conclude that $v = \langle 0; 5, 10, 14, 18, 22, 26, \dots \rangle$.

XII. (a) Suppose that $\alpha = \langle 0; 2b-1, \xi \rangle$ where $b \geq 2$ is an integer and $\xi \geq 2$ is a real number. Prove that $2\alpha = \langle 0; b-1, 1, 1 + 2/(\xi - 1) \rangle$.

(b) Suppose that $\alpha = \langle 0; 2b_1 - 1, 2b_2, 2b_3, 2b_4, \dots \rangle$ where each $b_j \geq 2$ is an integer. Prove that $2\alpha = \langle 0; b_1 - 1, 1, 1, b_2 - 1, 1, 1, b_3 - 1, 1, 1, b_4 - 1, 1, 1, \dots \rangle$.

(c) Prove what is possibly the coolest continued fraction expansion ever (Euler, 1737):

$$e = \langle 2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, \dots \rangle.$$