

Math 437/537 Homework #6

due Wednesday, November 26, 2008 at the beginning of class

- I. Some straightforward calculations—just write down the answers (for this one problem, I don't even care if you show your work). I know they're phrased in a strange way; humor me.
- (a) Let $y = 365.242199\dots$. Calculate the first 5 convergents of $y - 365$.
 - (b) Let $m = 29.53059\dots$. Calculate the first convergent of $\frac{y}{m} - 12$ that has a "score" of at least 5.
 - (c) Let $r = (\log \frac{3}{2})/(\log 2)$. Calculate *all* the convergents of r until you reach one that has a "score" of at least 5.
- II. Calculate the real number whose continued fraction is $\langle 4; 3, 2, 1, 3, 2, 1, 3, 2, 1, \dots \rangle$. Using the Quadratic Irrational Process, calculate the continued fraction expansion of $1 + (\sqrt{3}/5)$.
- III. Suppose that two real numbers s and t have continued fractions that agree up to and including a_n . Prove that $|s - t| < 1/k_n^2$. (Note that it doesn't matter whether we take the k_n belonging to s or t in this situation.)
- IV. Niven, Zuckerman, and Montgomery, Section 7.6, p. 344, #3 and #4. Note that c is a real number, not necessarily an integer.
- V. Niven, Zuckerman, and Montgomery, Section 7.8, p. 356, #1. Note that the problem says "all solutions" in two places where it means "all positive solutions".
- VI. Two quick, unrelated problems:
- (a) Of the first 1000 multiples of e , which one is closest to an integer? (Don't use brute force!)
 - (b) Suppose that d is a positive integer, not a square, that has a prime factor $p \equiv 3 \pmod{4}$. Prove that the period of the continued fraction expansion of \sqrt{d} is even.
- VII. (a) Given an integer $k \geq 2$, compute the continued fraction expansions of $\sqrt{k^2 - 1}$ and $\sqrt{k^2 + 4}$.
- (b) Suppose k is odd. Prove that the equation $x^2 + 1 = (k^2 + 4)y^2$ has infinitely many integer solutions (x, y) , and find the smallest positive solution (in terms of k).
- VIII. (a) Note that $x = 35, y = 6$ is a solution to the equation $x^2 - 34y^2 = 1$. Prove, without computing any continued fractions, that the equation $x^2 - 34y^2 = -1$ has no integer solutions.
- (b) Note that $x = \frac{5}{3}, y = \frac{1}{3}$ and $x = \frac{3}{5}, y = \frac{1}{5}$ are both solutions to the equation $x^2 - 34y^2 = -1$. Prove that the congruence $x^2 - 34y^2 \equiv -1 \pmod{m}$ has solutions for every positive integer m .

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IX. In this problem and the next, “smallest solution” means the ordered pair (x, y) of positive integers with y as small as possible.

(a) Find the continued fraction expansion of $\sqrt{61}$. You can just write down the answer without showing your work.

(b) Find the smallest solution to $x^2 - 61y^2 = -1$.

(c) Find the smallest solution to $x^2 - 61y^2 = 1$. (Hint: derive it directly from your answer to part (b). The point of this problem is that the answer is rather large, yet you can find it by hand pretty easily!)

X. Find the three smallest solutions to $x^2 - 73y^2 = 3$ without calculating any convergents beyond r_6 . (Remark: these three solutions involve integers x and y with at most nine digits. If you think the third-smallest solution involves 15- and 16-digit numbers, try again.)

XI. Find, with proof, a recursive definition of a sequence $\{x_n\}$ of distinct positive integers with the following property: for each n , there exists an integer z_n such that $(x_n, x_n + 1, z_n)$ is a Pythagorean triple.

XII. In Triangle Town, the streets are named 2nd Avenue, 3rd Avenue, 4th Avenue, and so on. There are exactly n houses on n th Avenue, the numbers of which (from west to east) are $1, 2, \dots, n$. Alice, Bob, and Eve have the following conversation:

BOB: What street do you live on, Alice?

(Alice whispers the answer to Bob so that Eve does not hear)

BOB: What’s the number of your house, Alice?

ALICE: I won’t tell you the number, but I will tell you how to find my house. Start at one end of my street and walk down it, adding up the numbers of all the houses one by one as you come to them. When your running total equals the following number ... *(Alice whispers a number to Bob so that Eve does not hear)* ... then my house is the last house you added in. *(Alice starts to walk away)*

BOB: Wait, Alice: which end of your street do I start from, the west end or the east end?

ALICE: That’s the best part: it doesn’t matter! *(Alice leaves)*

EVE: I know you don’t want to tell me Alice’s address, Bob, but at least tell me this much: how many digits are there in her street number?

(Bob answers Eve’s question)

What is Alice’s address (street and house number) if Bob’s answer was “One digit”? if Bob’s answer was “Two digits”? if Bob’s answer was “Three digits”? (Hint: complete the square.)