Thursday, December 5

Today we'll look at how the convergents h_j/k_j to a real number ξ are ~~good~~ the best rational approximations ξ.

Theorem: Fix $n \geq 0$. If $a \in \mathbb{Z}$, $b \in \mathbb{N}$ satisfies $|b\xi - a| < |k_n \xi - h_n|$, then $b \geq k_{n+1}$.

Interpretation: In the "how close to an integer is $k\xi$" game, the world-record setters are precisely $k_1\xi = \xi$, $k_2\xi$, $k_3\xi$, etc.

Exercise: verify that $|k_{n+1}\xi - h_{n+1}| < |k_n\xi - h_n|$.

Proof by contradiction: Suppose that $|b\xi - a| < |k_n\xi - h_n|$ but $b < k_{n+1}$. Define $x, y \in \mathbb{Z}$ by

$$x = k_{n+1} a - h_{n+1} b$$
$$y = -k_n a + h_n b.$$

Observation 1: $x \neq 0$ and $y \neq 0$.

- If $y = 0$ then $\frac{a}{b} = \frac{h_n}{k_n}$, so (since $(h_n, k_n) = 1$) $a = mh_n$ and $b = mk_n$ — but then $|b\xi - a| = m|k_n\xi - h_n|$ $\divideontimes$.
- If $x = 0$ then $\frac{a}{b} = \frac{h_{n+1}}{k_{n+1}} \Rightarrow a = mh_{n+1}$, $b = mk_{n+1} \geq k_{n+1}$ $\divideontimes$.

Solving for b:

$$b = \frac{xk_n + yk_{n+1}}{h_n k_{n+1} - k_n h_{n+1}} = (-1)^{n+1}(xk_n + yk_{n+1}).$$

Observation 2: x and y have opposite signs. (Suppose n odd.) Certainly $x, y < 0$ is impossible; also, $x, y > 0 \Rightarrow$ RHS $\geq k_n + k_{n+1} > k_{n+1} > b$.

Now note that

$$x(k_n\xi - h_n) + y(k_{n+1}\xi - h_{n+1})$$
$$= (b\xi - a)(h_n k_{n+1} - k_n h_{n+1}) = (-1)^{n+1}(b\xi - a)$$

have the same sign. Consequently,

$$|b\xi - a| > |x(k_n\xi - h_n)| \geq |k_n\xi - h_n|. \quad \divideontimes$$

Corollary: Let $n \geq 0$. If $\left|\xi - \frac{a}{b}\right| < \left|\xi - \frac{h_n}{k_n}\right|$, then $b > k_n$.

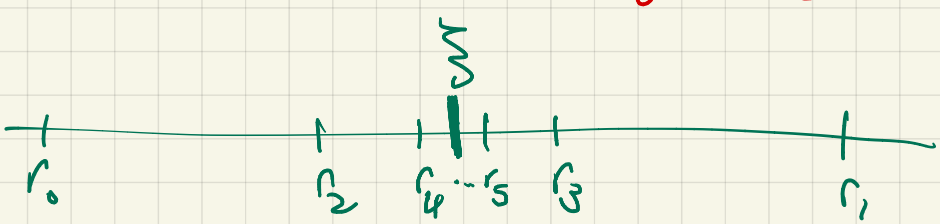
Proof by contradiction. Suppose $\left|\xi - \frac{a}{b}\right| < \left|\xi - \frac{h_n}{k_n}\right|$ but $b \leq k_n$. Multiplying:

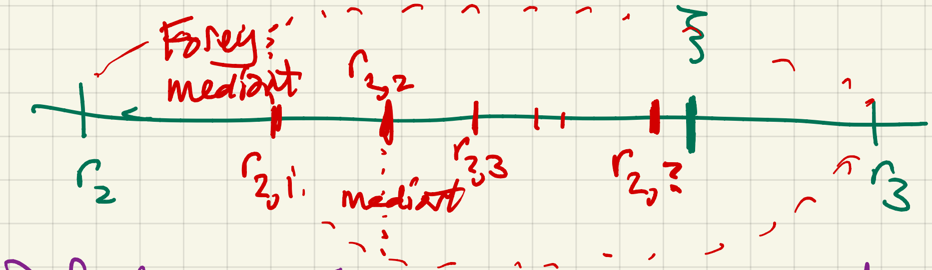
$$|b\xi - a| < |k_n \xi - h_n|$$

— but that implies $b \geq k_{n+1}$ by the theorem, $\times$.

Interpretation: In the "how close to $\frac{a}{b}$ to ξ" game, the convergents are world-record holders; but there can be others.

(I believe) that all other world record holders come from "secondary convergents":





Definition: The secondary convergents to ξ are the numbers

$$r_{j,i} = \frac{i h_{j+1} + h_j}{i k_{j+1} + k_j}, \quad 1 \leq i \leq a_{j+2} - 1.$$

Notes: $r_{j,0} = r_j$.

- $r_{j,i}$ is the mediant of $r_{j,i-1}$ and r_{j+1}.
- $r_{j,a_{j+2}} = \dfrac{a_{j+2} h_{j+1} + h_j}{a_{j+2} k_{j+1} + k_j} = \dfrac{h_{j+2}}{k_{j+2}}$.

Recall:

- Dirichlet's theorem: if $\xi \in \mathbb{R}\setminus\mathbb{Q}$, then there are infinitely many a/b such that $\left|\xi - \frac{a}{b}\right| < \frac{1}{b^2}$.
- If $\frac{h_j}{k_j}$ is a convergent of ξ, then $\left|\xi - \frac{h_j}{k_j}\right| < \frac{1}{k_j^2}$.

Semi-converse:

Theorem: If $\left|\xi - \frac{a}{b}\right| < \frac{1}{2b^2}$, then a/b is a convergent to ξ.

Note: WLOG $(a,b)=1$.

Proof by contradiction: Suppose a/b is not a convergent. Choose $j \in \mathbb{N}_0$ such that $k_j \le b < k_{j+1}$. By today Theorem,

$$|k_j\xi - h_j| \le |b\xi - a| = b\left|\xi - \frac{a}{b}\right| < \frac{1}{2b}.$$

$$\Rightarrow \left|\xi - h_j/k_j\right| < \frac{1}{2bk_j}.$$

Since $\frac{a}{b} \ne \frac{h_j}{k_j}$,

$$\frac{1}{bk_j} \le \left|\frac{a}{b} - \frac{h_j}{k_j}\right| \le \left|\frac{h_j}{k_j} - \xi\right| + \left|\xi - \frac{a}{b}\right| < \frac{1}{2bk_j} + \frac{1}{2b^2}.$$

$$\Rightarrow \frac{1}{k_j} < \frac{1}{2k_j} + \frac{1}{2b}$$

$$\frac{1}{2k_j} < \frac{1}{2b} \Rightarrow b < k_j, \quad \times.$$

Theorem (Hurwitz): If $\xi \in \mathbb{R}\setminus\mathbb{Q}$, there are infinitely many $\frac{a}{b}$ with $\left|\xi - \frac{a}{b}\right| < \frac{1}{\sqrt{5}\,b^2}$.

- Several proofs; in NZM (Theorem 7.17), they show that one of r_j, r_{j+1}, r_{j+2} must satisfy this.

It turns out $\sqrt{5}$ is the best possible constant: the golden ratio $\frac{\sqrt{5}+1}{2}$ has only finitely many convergents with $\left|\frac{\sqrt{5}+1}{2} - \frac{h_j}{k_j}\right| < \frac{1}{(\sqrt{5}+\varepsilon)k_j^2}$.

It turns out: if we remove the countably many numbers of the form $\frac{a\phi+b}{c\phi+d}$ where $\phi = \frac{\sqrt{5}+1}{2}$, $a,b,c,d \in \mathbb{Z}$, then we can further improve the $\sqrt{5}$ to $\sqrt{8}$.

- $\sqrt{2}$ shows that this is now best possible;
- if we remove $\frac{a\sqrt{2}+b}{c\sqrt{2}+d}$, we can improve $\sqrt{8}$ to $\frac{1}{5}\sqrt{221}$; ...

related to "Lagrange spectrum" and "Markov spectrum".

Let's connect continued fractions to ergodic theory (dynamical systems).

Let $T: (1,\infty) \to (1,\infty)$ by $T(x) = \frac{1}{\{x\}}$ (fractional part), so that $\xi_{j+1} = T(\xi_j)$ in The Process.

(technically $(1,\infty) \setminus \mathbb{Q}$)

T has an "invariant measure" on $(1,\infty)$:

$$d\mu(x) = \frac{1}{\log 2} \frac{1}{x+x^2}\, dx.$$

This means: if $S \subset (1,\infty)$ is open, then

$$\int_S d\mu(x) = \int_{T^{-1}(S)} d\mu(x).$$

Consequently, for almost all $\xi \in (1,\infty)$, the iterates $T(\xi), T(T(\xi)), \ldots$ are equidistributed with respect to $d\mu(x)$.

In particular, the proportion of j such that $\xi_j \in [m, m+1)$ is

$$\int_m^{m+1} \frac{1}{\log 2} \frac{1}{x+x^2}\,dx = \frac{1}{\log 2} \log\left(1+\frac{1}{m(m+2)}\right).$$

Exercise: Show that $\sum_{m=1}^{\infty} \frac{1}{\log 2} \log\left(1+\frac{1}{m(m+2)}\right) = 1.$

m	1	2	3	4	5
$\frac{1}{\log 2} \log\left(1+\frac{1}{m(m+2)}\right)$	.415	.170	.093	.059	.041

☆

Consequently, for almost all real numbers ξ, approximately 41.5% of its partial quotients a_j equal 1, ☆ 17.0% equal 2, 9.3% equal 3, and so on.

Exceptions: $\mathbb{Q}$, quadratic irrationals, e, $\frac{ae+b}{ce+d}$.

Conjectured successes: π, algebraic #s of degree ≥ 3, everything else we've ever heard of.

Observation:

$a_2 a_3 a_4 \cdots a_j \leq k_j \leq (a_2+1)(a_3+1)\cdots(a_j+1)$.

So statistics of the $a_j \rightsquigarrow$ statistics of the k_j, perhaps $k_j^{1/j}$.

Theorem: (Khinchin)

For almost all $\xi \in \mathbb{R}$,

$$\lim_{j\to\infty} k_j^{1/j} = C \quad \text{where}$$

$$C = e^{\pi^2/12\log 2} \approx 3.2758.$$