Thursday, November 14

Recall:

Lemma: If $\frac{a}{b}, \frac{c}{d}$ are distinct rational numbers, then $\left|\frac{a}{b}-\frac{c}{d}\right| \geq \frac{1}{|bd|}$. ☆

Theorem (Dirichlet) Let $x \in \mathbb{R}$ and $n \in \mathbb{N}$. There exists $\frac{a}{b} \in \mathbb{Q}$ with $1 \leq b \leq n$ such that

$$\left|x - \frac{a}{b}\right| \leq \frac{1}{b(n+1)}.$$

Corollary: If $x \in \mathbb{R} \setminus \mathbb{Q}$, then there are infinitely many distinct rational numbers $\frac{a}{b}$ such that $\left|x - \frac{a}{b}\right| < \frac{1}{b^2}$.

Remark: If $x = \frac{c}{d}$ were rational, then $\left|x - \frac{a}{b}\right| < \frac{1}{b^2}$ is impossible when $b > d$, by Lemma ☆.

Proof: For every $n \in \mathbb{N}$, Dirichlet's theorem gives us some $\frac{a}{b}$ with $b \leq n$ and

$$\left|x - \frac{a}{b}\right| \leq \frac{1}{b(n+1)} < \frac{1}{b}\frac{1}{b} = \frac{1}{b^2}.$$

Why does this result in infinitely many distinct $\frac{a}{b}$? Given $\frac{a}{b}$, the inequality $\left|x - \frac{a}{b}\right| \leq \frac{1}{b(n+1)}$ is impossible once

$$n+1 > \frac{1}{b\left|x - \frac{a}{b}\right|} = \frac{1}{|bx - a|}. \longleftarrow \text{here we use } x \notin \mathbb{Q}.$$

//

Lemma: Let $p(x) \in \mathbb{Z}[x]$ have degree d, and let $\frac{a}{b} \in \mathbb{Q}$. If $p\left(\frac{a}{b}\right) \neq 0$, then

$$\left|p\left(\frac{a}{b}\right)\right| \geq \frac{1}{b^d}.$$

Remark: the $d=1$ case is Lemma ☆:

$p(x) = nx - m$, and $\left|p\left(\frac{a}{b}\right)\right| = \left|\frac{na}{b} - m\right| \geq \frac{1}{b^1}$.

$$\Rightarrow \left|\frac{a}{b} - \frac{m}{n}\right| \geq \frac{1}{bn}.$$

Lemma: Let $p(X) \in \mathbb{Z}[X]$ have degree d, and let $\frac{a}{b} \in \mathbb{Q}$. If $p(\frac{a}{b}) \neq 0$, then $|p(\frac{a}{b})| \geq \frac{1}{b^d}$.

Proof: Let $p(X) = c_d X^d + c_{d-1} X^{d-1} + \cdots + c_2 X^2 + c_1 X + c_0$, where $c_j \in \mathbb{Z}$ and $c_d \neq 0$. Then

$$b^d p(\tfrac{a}{b}) = c_d a^d + c_{d-1} a^{d-1} b + \cdots + c_2 a^2 b^{d-2} + c_1 a b^{d-1} + c_0 b^d \in \mathbb{Z} \setminus \{0\}.$$

Thus $|b^d p(\frac{a}{b})| \geq 1$. //

Definition: Let $\alpha \in \mathbb{R}$. We say α is <u>algebraic of degree d</u> if there exists an irreducible polynomial $p(X) \in \mathbb{Z}[X]$ of degree d such that $p(\alpha) = 0$. If α is not algebraic (to any degree $d \geq 1$), we call α <u>transcendental</u>.

Examples:

- "Algebraic of degree 1" is the same as "rational": $\frac{a}{b}$ is a root of $p(X) = bX - a$.
- $\sqrt{537}$ is algebraic of degree 2: $p(X) = X^2 - 537$.
- $\sqrt[3]{4 + \sqrt[5]{6}}$ is algebraic of degree 15: $p(X) = (X^3 - 4)^5 - 6$. ← if this is irreducible...
- Not all algebraic numbers can be written with "nested radicals" — Galois theory.

Exercise: Prove that the degree of an algebraic number is unique. Hint: if $a(X), b(X) \in \mathbb{Z}[X]$ irreducible, $\deg b > \deg a$, and $a(\alpha) = 0 = b(\alpha)$, then write $b(X) = a(X)q(X) + r(X)$ with $\deg r < \deg a$.

Theorem (Liouville, 1844): Let α be algebraic of degree d. There exists some constant $C(\alpha) > 0$ such that for any $\frac{a}{b} \in \mathbb{Q}$, $\frac{a}{b} \neq \alpha$, we have $\left|\alpha - \frac{a}{b}\right| \geq \frac{C(\alpha)}{b^d}$.

Remark: Lemma ☆ is the case $d = 1$.

Proof: If we restrict $C(\alpha) \leq 1$, then we only have to look at $\frac{a}{b}$ with $\left|\alpha - \frac{a}{b}\right| \leq 1$.

Let $p(X) \in \mathbb{Z}[X]$ be irreducible of degree d with $p(\alpha) = 0$; then $p\left(\frac{a}{b}\right) \neq 0$ (since if $p\left(\frac{a}{b}\right) = 0$ then $p(X) = (bX - a)q(X)$).

Then
$$p\left(\frac{a}{b}\right) = p\left(\frac{a}{b}\right) - p(\alpha) = \left(\frac{a}{b} - \alpha\right) p'(t)$$
for some t between α and $\frac{a}{b}$ (Mean Value Theorem).

Take $C(\alpha) = \min\left\{1, \ 1/\max\{|p'(t)| : t \in [\alpha - 1, \alpha + 1]\}\right\}$.

We conclude from the previous lemma that
$$\frac{1}{b^d} \leq \left|p\left(\frac{a}{b}\right)\right| = \left|\alpha - \frac{a}{b}\right| |p'(t)|$$
$$\leq \left|\alpha - \frac{a}{b}\right| \frac{1}{C(\alpha)}. \quad //$$

Corollary: Transcendental numbers exist!

Proof (Liouville): Choose
$$\alpha = \sum_{n=1}^{\infty} 10^{-n!} = 0.110001\,000000\,000000\,000001\ldots$$
($1!$, $2!$, $3!$, $4!$)

Let $\frac{a_k}{b_k} = \sum_{n=1}^{k} 10^{-n!}$, so $b_k = 10^{k!}$. Then
$$\left|\alpha - \frac{a_k}{b_k}\right| = \sum_{n=k+1}^{\infty} 10^{-n!}.$$
Each summand is at most $\frac{1}{2}$ the previous summand, so
$$\left|\alpha - \frac{a_k}{b_k}\right| \leq \sum_{n=k+1}^{\infty} 10^{-(k+1)!} \left(\frac{1}{2}\right)^{n-(k+1)} = \frac{2}{10^{(k+1)!}}$$
$$= \frac{2}{(10^{k!})^{k+1}} = \frac{2}{b_k^{k+1}}.$$

If α were of degree d, then
$$\frac{C(\alpha)}{b_k^d} \leq \left|\alpha - \frac{a_k}{b_k}\right| \leq \frac{2}{b_k^{k+1}},$$
impossible for k large. //

Remark: It's not hard to show that the set of algebraic numbers is countable; this is another proof that transcendental numbers exist (and are plentiful).

But Cantor didn't prove his stuff until decades ~~after~~ Liouville.

Various improvements of the "d" in $\frac{C(\alpha)}{b^d}$ over the years... culminating in:

Theorem (Roth, 1955): Let α be algebraic of degree $d \geq 2$. For any $\varepsilon > 0$, there exists $C(\alpha, \varepsilon) > 0$ such that

$$\left|\alpha - \frac{a}{b}\right| \geq \frac{C(\alpha,\varepsilon)}{b^{2+\varepsilon}}.$$

(Note: if $d = 2$ then ε isn't necessary)

Definition: The Farey sequence of order n, denoted $\mathcal{F}_n$ [\mathcal F_n], is the ordered sequence of reduced proper fractions with denominator at most n.

Example: $\mathcal{F}_5 = \left\{\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}\right\}$

Question: How to find the neighbours of $\frac{a}{b} \in \mathcal{F}_n$?

Proposition: If $\frac{a}{b} \in \mathcal{F}_n$, then the next element of $\mathcal{F}_n$ is $\frac{x}{y}$, where $y \equiv -a^{-1} \pmod{b}$ and $n - b < y \leq n$, and $x = \frac{ay+1}{b}$.

Proof: First, $ay + 1 \equiv a(-a^{-1}) + 1 \equiv 0 \pmod{b}$, so $x \in \mathbb{Z}$. And $y \leq n$, so $\frac{x}{y} \in \mathcal{F}_n$; and $\frac{x}{y} = \frac{ay+1}{by} > \frac{ay}{by} = \frac{a}{b}$.

Now suppose that $\frac{c}{d} \in \mathcal{F}_n$ satisfies

$$\frac{a}{b} < \frac{c}{d} < \frac{x}{y}.$$

Then:

$$\left(\frac{c}{d}-\frac{a}{b}\right)+\left(\frac{x}{y}-\frac{c}{d}\right)=\frac{x}{y}-\frac{a}{b}$$

$$=\frac{bx-ay}{yb} = \frac{1}{yb}, \text{ by definition of } x.$$

However, by Lemma A

$$\left(\frac{c}{d}-\frac{a}{b}\right)+\left(\frac{x}{y}-\frac{c}{d}\right) \geq \frac{1}{bd}+\frac{1}{dy}$$

$$=\frac{y+b}{bdy} \geq \frac{n+1}{bdy} = \frac{1}{by}\,\frac{n+1}{d} > \frac{1}{by}$$

$\Rightarrow$ contradiction. //

Corollary 1: If $\frac{a}{b}<\frac{x}{y}$ are consecutive in $\mathcal{F}_n$, then $bx-ay=1$.

Corollary 2: If $\frac{a}{b}<\frac{c}{d}<\frac{x}{y}$ are consecutive in $\mathcal{F}_n$, then $\frac{c}{d}$ equals the "Farey mediant" $\frac{c}{d}=\frac{a+x}{b+y}$.

Proof: From Corollary 1, $bc-ad=1$ and $dx-cy=1$ which imply $bc-ad=dx-cy$ and so

$bc+cy=ad+dx$

$c(b+y)=d(a+x)$. ✓

Corollary 3: If $\frac{a}{b}<\frac{x}{y}$ are consecutive in $\mathcal{F}_n$, then $b+y \geq n+1$.

(or else $\frac{c}{d}=\frac{a+x}{b+y}$ would be in $\mathcal{F}_n$).

We can reprove Dirichlet's theorem using Farey sequences (!).

Theorem (Dirichlet) Let $x \in \mathbb{R}$ and $n \in \mathbb{N}$.
There exists $\frac{a}{b} \in \mathbb{Q}$ with $1 \le b \le n$ such that
$$\left|x - \frac{a}{b}\right| \le \frac{1}{b(n+1)}.$$

Proof: Without loss of generality, $x \in [0,1]$.
Consider $\mathcal{F}_n$ with the given n.

- If $x \in \mathcal{F}_n$, take $\frac{a}{b} = x$.
- If $x \notin \mathcal{F}_n$, then choose $\frac{c}{d} < \frac{e}{f}$ consecutive in $\mathcal{F}_n$ with
$$\frac{c}{d} < x < \frac{e}{f}.$$

Case 1: Suppose $x \le \frac{c+e}{d+f}$. Then set $\frac{a}{b} = \frac{c}{d}$.

Check:
$$\left|x - \frac{a}{b}\right| = x - \frac{c}{d} \le \frac{c+e}{d+f} - \frac{c}{d}$$
$$= \frac{\cancel{cd} + de - \cancel{cd} - cf}{d(d+f)} = \frac{de-cf}{d(d+f)} = \frac{1}{d(d+f)}$$

by Corollary 1; and $d+f \ge n+1$ by Corollary 3, so
$$\left|x - \frac{a}{b}\right| \le \frac{1}{d(d+f)} \le \frac{1}{d(n+1)}.$$

Case 2: Suppose $x \ge \frac{c+e}{d+f}$.
Then choose $\frac{a}{b} = \frac{e}{f}$
(essentially the same proof).