Tuesday, November 19

Notation for the Euclidean algorithm, applied to the pair of integers u_0 and u_1:

$$u_0 = u_1 a_0 + u_2 \qquad (0 < u_2 < u_1)$$
$$u_1 = u_2 a_1 + u_3 \qquad (0 < u_3 < u_2)$$
$$\vdots$$

(At some point, say, $u_{k+1} = 0$.)

We call the a_j the partial quotients.

Equivalently, $\dfrac{u_0}{u_1} = a_0 + \dfrac{1}{u_1/u_2}$ note: $a_0 = \left\lfloor \dfrac{u_0}{u_1} \right\rfloor$.

$\dfrac{u_1}{u_2} = a_1 + \dfrac{1}{u_2/u_3}$. note: $a_1 = \left\lfloor \dfrac{u_1}{u_2} \right\rfloor$

$\vdots$

Recall the fractional part function $\{y\} = y - \lfloor y \rfloor$. Then (for example):

$$\frac{u_1}{u_2} = 1 \Big/ \left\{\frac{u_0}{u_1}\right\} = \frac{1}{\frac{u_0}{u_1} - a_0}.$$

This produces the continued fraction

$$\frac{u_0}{u_1} = \langle a_0; a_1, a_2, \dots, a_k \rangle = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_k}}}}.$$

The Process.

Given $\xi \in \mathbb{R}$, define $\xi_0 = \xi$ and set

$a_0 = \lfloor \xi_0 \rfloor$, $\xi_1 = 1/\{\xi_0\}$;

$a_1 = \lfloor \xi_1 \rfloor$, $\xi_2 = 1/\{\xi_1\}$;

etc.

- If ξ is rational, then this process terminates (some $\xi_j \in \mathbb{Z}$), giving a finite continued fraction equaling ξ_0.
- If ξ is irrational, then this process never terminates, giving an "infinite continued fraction" $\langle a_0; a_1, a_2, \dots \rangle$.

Example: Let $\xi = \sqrt[3]{2} \approx 1.25992\ldots$

We have $\xi_0 = \xi$, and

$$a_0 = \lfloor \sqrt[3]{2} \rfloor = 1, \quad \xi_1 = \frac{1}{\xi_0 - 1} \approx 3.84732$$

$$a_1 = \lfloor \xi_1 \rfloor = 3, \quad \xi_2 = \frac{1}{\xi_1 - 3} \approx 1.18019$$

$$a_2 = \lfloor \xi_2 \rfloor = 1, \quad \xi_3 = \frac{1}{\xi_2 - 1} \approx 5.54974$$

$$a_3 = \lfloor \xi_3 \rfloor = 5, \quad \xi_4 = \frac{1}{\xi_3 - 5} \approx 1.81965.$$

Thus

$$\sqrt[3]{2} = \xi_0 = \langle a_0; a_1, a_2, a_3, \xi_4 \rangle$$

$$= \langle 1; 3, 1, 5, \xi_4 \rangle$$

$$= \frac{29\xi_4 + 5}{23\xi_4 + 4}.$$

Solving for ξ_4:

$$\xi_4 = \frac{4\sqrt[3]{2} - 5}{-23\sqrt[3]{2} + 29}.$$

[Shortcut rule: if $y = \dfrac{ax+b}{cx+d}$, then $x = \dfrac{dy-b}{-cy+a}$.

– like inverting matrices $\begin{pmatrix} \searrow & - \\ - & \nwarrow \end{pmatrix}$.]

- Some people use $((x))$ for the fractional part function.