# Thursday, November 21

RECALL

**The Process.**

Given $\xi \in \mathbb{R}$, define $\xi_0 = \xi$ and set

$$a_0 = \lfloor \xi_0 \rfloor, \quad \xi_1 = 1/\{\xi_0\};$$

$$a_1 = \lfloor \xi_1 \rfloor, \quad \xi_2 = 1/\{\xi_1\};$$

etc.

The Process gives us a sequence $(a_j)_{j=0}^{\infty}$ of partial quotients of $\xi$. Question:

if $\xi$ is irrational, does it make sense to say $\xi = \langle a_0; a_1, a_2, \dots \rangle$?

– The only sensible way to define the RHS is $\langle a_0; a_1, a_2, \dots \rangle = \lim_{n\to\infty} \langle a_0; a_1, \dots, a_n \rangle$.

Does the limit exist? If so, does it equal $\xi$?

TeX note: use \langle and \rangle for the delimiters $\langle$ and $\rangle$.

Notation: Given $a_0 \in \mathbb{Z}$ and $a_1, a_2, \dots \in \mathbb{N}$, recursively define $(h_j)$ and $(k_j)$ as:

$$h_{-2} = 0, \quad h_{-1} = 1, \quad h_j = a_j h_{j-1} + h_{j-2} \quad (j \geq 0)$$

$$k_{-2} = 1, \quad k_{-1} = 0, \quad k_j = a_j k_{j-1} + k_{j-2} \quad (j \geq 0),$$

Also define $r_j = h_j / k_j$ for $j \geq 0$.

Example: Using $\langle 1; 3, 1, 5, \dots \rangle$ coming from $\xi = \sqrt[3]{2}$:

| $j$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|
| $a_j$ | | | $1$ | $3$ | $1$ | $5$ |
| $h_j$ | $0$ | $1$ | $1$ | $4$ | $5$ | $29$ |
| $k_j$ | $1$ | $0$ | $1$ | $3$ | $4$ | $23$ |
| $r_j$ | $(0)$ | $(\infty)$ | $1$ | $4/3$ | $5/4$ | $29/23$ |

familiar numbers

Proposition: For any $x > 0$,

$$\langle a_0; a_1, \dots, a_{j-1}, x \rangle = \frac{x h_{j-1} + h_{j-2}}{x k_{j-1} + k_{j-2}}.$$

**Proposition:** For any $x>0$,

$$\langle a_0; a_1, \dots, a_{j-1}, x\rangle = \frac{x h_{j-1} + h_{j-2}}{x k_{j-1} + k_{j-2}}.$$

In particular, $\langle a_0; a_1, \dots, a_j\rangle$

$$= \frac{a_j h_{j-1} + h_{j-2}}{a_j k_{j-1} + k_{j-2}} = \frac{h_j}{k_j}.$$

**Proof:** by induction on $j \geq 0$.

$j=0$: $\langle a_0; a_1, \dots, a_{j-1}, x\rangle = \langle x\rangle = x$

$$= \frac{1x+0}{0x+1} = \frac{x h_{-1} + h_{-2}}{x k_{-1} + k_{-2}}. \checkmark$$

Suppose the proposition is true for $j$.

$\langle a_0; a_1, \dots, a_{j-1}, a_j, x\rangle$

$$= \langle a_0; a_1, \dots, a_{j-1}, a_j + \tfrac{1}{x}\rangle \quad \text{(ind. hyp.)}$$

$$= \frac{(a_j + 1/x) h_{j-1} + h_{j-2}}{(a_j + 1/x) k_{j-1} + k_{j-2}}$$

$$= \frac{x(a_j h_{j-1} + h_{j-2}) + h_{j-1}}{x(a_j k_{j-1} + k_{j-2}) + k_{j-1}} = \frac{x h_j + h_{j-1}}{x k_j + k_{j-1}}. \checkmark$$

**Definition:** Given $\xi \in \mathbb{R}$, if The Process on $\xi$ to get $a_0, a_1, \dots$, then the rational number $\frac{h_j}{k_j} = r_j$ is the $j^{th}$ convergent to $\xi$.

$= \langle a_0; a_1, \dots, a_j\rangle$.

**Example:** The $0^{th}$ to $3^{rd}$ convergents of $\sqrt[3]{2}$ are $1, \frac{4}{3}, \frac{5}{4}, \frac{29}{23}$. Note $\sqrt[3]{2} \approx 1.25992$ while the convergents are $1.0$, $1.333\ldots$, $1.25$, $1.26087\ldots$ getting closer to $\sqrt[3]{2}$.

**Example:** Take $\xi = \frac{\sqrt{5}+1}{2}$ (golden ratio).

The Process gives $a_0 = \left\lfloor \frac{\sqrt{5}+1}{2} \right\rfloor = 1$ and

$$\xi_1 = \frac{1}{\{\xi\}} = \frac{1}{\frac{\sqrt{5}+1}{2} - 1} = \frac{1}{\frac{\sqrt{5}-1}{2}} \frac{\sqrt{5}+1}{\sqrt{5}+1} = \frac{\sqrt{5}+1}{2}$$

$$= \xi.$$

So $a_j = 1$ and $\xi_j = \xi$ for all $j \geq 0$;

$$\frac{\sqrt{5}+1}{2} = \langle 1; 1, 1, 1, \dots\rangle.$$

Moreover, (check) $h_j = F_{j+1}$ and $k_j = F_j$ where $F_j$ is the $j^{th}$ Fibonacci number.

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[Note: $F_0 = 0$, $F_1 = 1$, $F_2 = 1$, $F_j = F_{j-1} + F_{j-2}$.
Makes nice things true; example: $a \mid b \Rightarrow F_a \mid F_b$.]

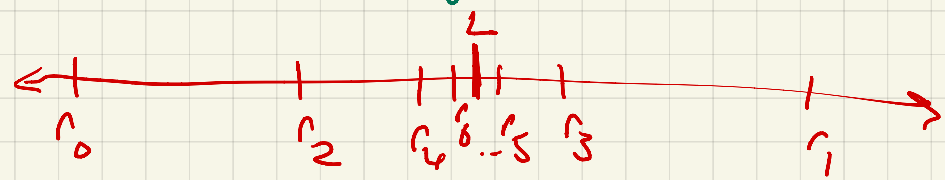
So the $j^{th}$ convergent to $\frac{\sqrt{5}+1}{2}$ is $\frac{F_{j+1}}{F_j}$,
and indeed $\lim\limits_{j\to\infty} \frac{F_{j+1}}{F_j} = \frac{\sqrt{5}+1}{2}$.

Sidenote: if $a_0, a_1, \ldots \in \mathbb{N}$, it's easy to show that $h_j, k_j \geq F_j$. So the $h_j$ and $k_j$ grow at least exponentially for any $\xi$.

Lemma: For $j \geq -1$, we have
$$h_j k_{j-1} - k_j h_{j-1} = (-1)^{j-1}.$$
In particular, $(h_j, k_j) = 1$, and $r_j - r_{j-1} = \frac{(-1)^{j-1}}{k_j k_{j-1}}$.

Proof: Exercise.

Nice consequence:
$$r_j = r_0 + \sum_{i=1}^{j} (r_i - r_{i-1}) = r_0 + \sum_{i=1}^{j} \frac{(-1)^{i-1}}{k_i k_{i-1}}$$
passes the alternating series test and hence converges ($\lim\limits_{j\to\infty} r_j$ exists)!




$r_0 < r_2 < r_4 < \cdots \quad \cdots < r_5 < r_3 < r_1$.

Turns out: limit is indeed $\xi$.

Theorem ("convergence of convergents"): Let $\xi \in \mathbb{R} \setminus \mathbb{Q}$, and let $a_0, a_1, \ldots$ be its partial quotients; let $\xi_j, h_j, k_j, r_j$ be as defined above. Then
$$\xi - r_j = \frac{(-1)^j}{k_j(\xi_{j+1} k_j + k_{j-1})}.$$
In particular, $\lim\limits_{j\to\infty} r_j = \xi$.

Proof: By The Process and the earlier Proposition,

$$\xi - r_j = \langle a_0; a_1, \ldots, a_j, \xi_{j+1} \rangle - r_j$$

$$= \frac{\xi_{j+1} h_j + h_{j-1}}{\xi_{j+1} k_j + k_{j-1}} - \frac{h_j}{k_j}$$

$$= \frac{(\cancel{\xi_{j+1} h_j k_j} + h_{j-1} k_j) - (\cancel{\xi_{j+1} k_j h_j} + k_{j-1} h_j)}{k_j(\xi_{j+1} k_j + k_{j-1})}$$

and the numerator is

$$h_{j-1} k_j - k_{j-1} h_j = -(-1)^{j-1} = (-1)^j$$

by the Lemma. ✓

How good are these rational approximations?

$$|\xi - r_j| = \frac{1}{k_j(\xi_{j+1} k_j + k_{j-1})}.$$

The continued fraction of $\xi_{j+1}$ is $\langle a_{j+1}; a_{j+2}, \ldots \rangle$ (by The Process). In particular,

★ $(a_{j+1} \leq \xi_{j+1} \leq a_{j+1} + 1)$. Therefore:

(1) Given $n \in \mathbb{N}$, choose $j \geq 0$ with $k_j \leq n < k_{j+1}$; then

$$|\xi - r_j| = \left|\xi - \frac{h_j}{k_j}\right| = \frac{1}{k_j(\xi_{j+1} k_j + k_{j-1})}$$

$$\leq \frac{1}{k_j(a_{j+1} k_j + k_{j-1})} = \frac{1}{k_j k_{j+1}} \leq \frac{1}{k_j(n+1)}.$$

Thus every convergent to $\xi$ is an instance of Dirichlet's box principle.

(2) $$\left|\xi - \frac{h_j}{k_j}\right| = \frac{1}{k_j^2} \cdot \frac{1}{\xi_{j+1} + k_{j-1}/k_j} = \frac{1}{\theta k_j^2}$$

where $\theta = \xi_{j+1} + \underbrace{k_{j-1}/k_j}_{\in(0,1)}$ satisfies

$$1 \leq a_{j+1} \leq \theta \leq a_{j+1} + 2. \quad \Leftarrow ★$$

So every $\frac{h_j}{k_j}$ satisfies $\left|\xi - \frac{h_j}{k_j}\right| < \frac{1}{k_j^2}$;

and large $a_j$ correspond to even better rational approximations $r_j$.

[Tangent.. $\left|\xi - \frac{a}{b}\right| < \frac{1}{b^2} \iff$

$|b\xi - a| < \frac{1}{b} \iff$ either $\{b\xi\} < \frac{1}{b}$

or $\{b\xi\} > 1 - \frac{1}{b}$. So

($\xi$ not having rational approximations much better than $1/b^2$) $\iff$

(multiples of $\xi$ aren't that close to integers)

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