Tuesday, November 26

Notation. Given $a_0 \in \mathbb{Z}$ and $a_1, a_2, \ldots \in \mathbb{N}$, recursively define (h_j) and (k_j) as:

$h_{-2} = 0,\ h_{-1} = 1,\quad h_j = a_j h_{j-1} + h_{j-2}\quad (j \geq 0)$

$k_{-2} = 1,\ k_{-1} = 0,\quad k_j = a_j k_{j-1} + k_{j-2}\quad (j \geq 0).$

[RECALL]

Recall, if $\xi = \langle a_0; a_1, \ldots, a_{j-1}, \xi_j \rangle$, then Thursday's Proposition tells us that

$$\xi = \frac{\xi_j h_{j-1} + h_{j-2}}{\xi_j k_{j-1} + k_{j-2}}.$$

It follows that

$$\xi_j = \frac{\xi k_{j-2} - h_{j-2}}{-\xi k_{j-1} + h_{j-1}}. \qquad \star$$

Example: Let $\xi = \xi_0 = 6+\sqrt{41}$

$\doteq 12.40312$

- $a_0 = \lfloor \xi_0 \rfloor = 12$, $\xi_1 = \dfrac{1}{\xi_0 - 12} \doteq 2.48062$
- $a_1 = \lfloor \xi_1 \rfloor = 2$, $\xi_2 = \dfrac{1}{\xi_1 - 2} \doteq 2.08062$
- $a_2 = \lfloor \xi_2 \rfloor = 2$, $\xi_3 = \dfrac{1}{\xi_2 - 2} \doteq 12.40312$

Hmmm!

j	-2	-1	0	1	2
a_j			12	2	2
h_j	0	1	12	25	62
k_j	1	0	1	2	5

Let's use ☆ to write ξ_1, ξ_2, ξ_3 in terms of ξ_0.

- $\xi_1 = \dfrac{\xi k_{-1} - h_{-1}}{-\xi k_0 + h_0} = \dfrac{(6+\sqrt{41})\,0 - 1}{-(6+\sqrt{41})\,1 + 12}$

$= -\dfrac{1}{6-\sqrt{41}} \cdot \dfrac{6+\sqrt{41}}{6+\sqrt{41}} = \dfrac{6+\sqrt{41}}{5}$,

- $\xi_2 = \dfrac{\xi k_0 - h_0}{-\xi k_1 + h_1} = \dfrac{(6+\sqrt{41})\,1 - 12}{-(6+\sqrt{41})\,2 + 25}$

$= \dfrac{-6+\sqrt{41}}{13-2\sqrt{41}} \cdot \dfrac{13+2\sqrt{41}}{13+2\sqrt{41}} = \dfrac{4+\sqrt{41}}{5}$

- $\xi_3 = \dfrac{\xi k_1 - h_1}{-\xi k_2 + h_2} = \dfrac{(6+\sqrt{41})\,2 - 25}{-(6+\sqrt{41})\,5 + 62}$

$= \dots = 6+\sqrt{41} = \xi_0$.

Consequence of $\xi_3 = \xi_0$: the CF is periodic! $6+\sqrt{41} = \langle 12; 2, 2, 6+\sqrt{41} \rangle$

$= \langle 12; 2, 2, 12, 2, 2, 6+\sqrt{41} \rangle$

$= \dots$

We can write

$$6+\sqrt{41} = \langle \overline{12; 2, 2} \rangle \text{ or } \langle 12; \overline{2, 2, 12} \rangle.$$

It follows that

$$\begin{aligned}\sqrt{41} &= (6+\sqrt{41}) - 6 \\ &= \langle 12; \overline{2, 2, 12} \rangle - 6 \\ &= \langle 6; \overline{2, 2, 12} \rangle.\end{aligned}$$

Compare to an earlier example:

$$\begin{aligned}\frac{\sqrt{5}+1}{2} &= \langle 1; 1, 1, \ldots \rangle \\ &= \langle 1; \overline{1} \rangle \qquad (\langle \overline{1} \rangle ?)\end{aligned}$$

Note both $\sqrt{41}$ & $\frac{\sqrt{5}+1}{2}$ are both quadratic irrationals (algebraic of degree 2).

Theorem: If the continued fraction of $\xi \in \mathbb{R}$ is eventually periodic, then ξ is a quadratic irrational.

Proof: (Note $\xi \notin \mathbb{Q}$ since its CF doesn't terminate.) Suppose that for some $0 \le m < n$,

$$\xi = \langle a_0; a_1, \ldots, a_{m-1}, \overline{a_m, \ldots, a_{n-1}} \rangle.$$

Then in particular, $\xi_m = \xi_n = \langle \overline{a_m; a_{m+1}, \ldots, a_{n-1}} \rangle$.

By ☆,

$$\frac{\xi k_{m-2} - h_{m-2}}{-\xi k_{m-1} + h_{m-1}} = \xi_m = \xi_n = \frac{\xi k_{n-2} - h_{n-2}}{-\xi k_{n-1} + h_{n-1}}.$$

$$\Rightarrow (\xi k_{m-2} - h_{m-2})(-\xi k_{n-1} + h_{n-1}) = (\xi k_{n-2} - h_{n-2})(-\xi k_{m-1} + h_{m-1})$$

$\Rightarrow \xi$ is a root of a quadratic polynomial / $\mathbb{Z}$. //