Math 437/537—Group Work #9

Tuesday, November 19, 2024

Definition: A **continued fraction** is an expression of the form

$$x_0 + \cfrac{1}{x_1 + \cfrac{1}{x_2 + \cfrac{1}{\ddots + \cfrac{1}{x_j}}}}, \tag{1}$$

where $x_0 \in \mathbb{R}$ and $x_1, \ldots, x_j > 0$. A shorthand notation for the expression (1) is $\langle x_0; x_1, x_2 \ldots, x_j \rangle$.

1. *Define the function* $f(x) = \langle 1; 3, 1, 5, x \rangle$ *for* $x > 0$.

   (a) *Show that* $f(x) = \left\langle 1; 3, 1, \frac{5x+1}{x} \right\rangle$.

   (b) *Part (a) gave a length-4 continued fraction that equals* $f(x)$*. Find a similar length-3 continued fraction that also equals* $f(x)$*; then, length-2; then, length-1.*

   (c) *Evaluate the rational number* $\langle 0; 3, 1, 5, 100 \rangle$.

   (d) *Verify the identities*

   $$\langle x_0; x_1, x_2, \ldots, x_j \rangle = x_0 + \frac{1}{\langle x_1; x_2, x_3, \ldots, x_j \rangle}$$

   and

   $$\langle x_0; x_1, x_2, \ldots, x_j \rangle = \left\langle x_0; x_1, x_2, \ldots, x_{j-2}, x_{j-1} + \frac{1}{x_j} \right\rangle.$$

   (a) Bracketing the last two terms together,

   $$f(x) = 1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{5 + \cfrac{1}{x}}}} = 1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{\left(5 + \cfrac{1}{x}\right)}}} = \left\langle 1; 3, 1, 5 + \frac{1}{x} \right\rangle,$$

   which is the desired answer since $5 + \frac{1}{x} = \frac{5x+1}{x}$.

   (b) Similarly,

$$\left\langle 1; 3, 1, \frac{5x+1}{x} \right\rangle = 1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{\frac{5x+1}{x}}}} = 1 + \cfrac{1}{3 + \cfrac{1}{1 + \frac{x}{5x+1}}} = 1 + \cfrac{1}{3 + \cfrac{1}{\frac{6x+1}{5x+1}}} = \left\langle 1; 3, \frac{6x+1}{5x+1} \right\rangle$$

   and

$$\left\langle 1; 3, \frac{6x+1}{5x+1} \right\rangle = 1 + \cfrac{1}{3 + \cfrac{1}{\frac{6x+1}{5x+1}}} = 1 + \cfrac{1}{3 + \frac{5x+1}{6x+1}} = 1 + \cfrac{1}{\frac{(18x+3)+(5x+1)}{6x+1}} = \left\langle 1; \frac{23x+4}{6x+1} \right\rangle$$

and

$$\left\langle 1; \frac{23x+4}{6x+1} \right\rangle = 1 + \frac{1}{\frac{23x+4}{6x+1}} = 1 + \frac{6x+1}{23x+4} = \frac{(23x+4)+(6x+1)}{23x+4} = \left\langle \frac{29x+5}{23x+4} \right\rangle.$$

(c) Since $f(x) = \langle 1; 3, 1, 5, x\rangle$, we see that $f(x) - 1 = \langle 0; 3, 1, 5, x\rangle$, and so by part (b),

$$\langle 0; 3, 1, 5, 100\rangle = f(100) - 1 = \frac{29 \cdot 100 + 5}{23 \cdot 100 + 4} - 1 = \frac{2905}{2304} - 1 = \frac{601}{2304}.$$

(d) The first identity follows from just considering the huge (outermost) denominator in equation (1) as a continued fraction in its own right; the second identity follows from considering the second-to-last denominator $x_{j-1} + \frac{1}{x_j}$ as a single real number, as we did for specific numbers in parts (a) and (b).

2.

(a) *Use the Euclidean algorithm to calculate* $(73, 26)$ *and, at the same time, to show that* $\frac{73}{26} = \left\langle 2; \frac{26}{21} \right\rangle = \left\langle 2; 1, \frac{21}{5} \right\rangle = \langle 2; 1, 4, 5\rangle$.

(b) *Find a continued fraction that equals* $-\frac{196}{71}$. *Remember that* $x_1, x_2, \dots$ *must be positive.*

(a) Since $73 = 2 \cdot 26 + 21$, we have $\frac{73}{26} = 2 + \frac{21}{26} = \langle 2; \frac{26}{21}\rangle$. Since $26 = 1 \cdot 21 + 5$, we have $\frac{26}{21} = 1 + \frac{5}{21}$ and so $\langle 2; \frac{26}{21}\rangle = \langle 2; 1, \frac{21}{5}\rangle$. Since $21 = 4 \cdot 5 + 1$, we have $\frac{21}{5} - 4 + \frac{1}{5}$ and so $\langle 2; 1, \frac{21}{5}\rangle = \langle 2; 1, 4, 5\rangle$. (These calculations also show that $(73, 26) = (26, 21) = (21, 5) = (5, 1) = 1$, but that ends up not mattering so much here.)

(b) Encouraged by part (a), we use the Euclidean algorithm, remembering that the division algorithm always returns a nonnegative remainder; so we must start by writing $-196 = -3 \cdot 71 + 17$, so that $-\frac{196}{71} = -3 + \frac{17}{71} = \langle -3; \frac{71}{17}\rangle$. From here it's more standard: $71 = 4 \cdot 17 + 3$, so $\frac{71}{17} = 4 + \frac{3}{17}$ and $\langle -3; \frac{71}{17}\rangle = \langle -3; 4, \frac{17}{3}\rangle$. Then $17 = 5 \cdot 3 + 2$, so $\frac{17}{3} = 5 + \frac{2}{3}$ and $\langle -3; 4, \frac{17}{3}\rangle = \langle -3; 4, 5, \frac{3}{2}\rangle$; then $3 = 1 \cdot 2 + 1$, so $\frac{3}{2} = 1 + \frac{1}{2}$ and $\langle -3; 4, 5, \frac{3}{2}\rangle = \langle -3; 4, 5, 1, 2\rangle$.

For problem #3, all continued fractions will have *integer* entries.

3. *Define* $\alpha = \langle x_0; x_1, \dots, x_j\rangle$ *and* $\beta = \langle y_0; y_1, \dots, y_k\rangle$.

(a) *Suppose* $x_0 \neq y_0$. *When is* $\alpha < \beta$*?*

(b) *Suppose* $x_0 = y_0$ *but* $x_1 \neq y_1$. *When is* $\alpha < \beta$*?*

(c) *Suppose* $x_0 = y_0$ *and* $x_1 = y_1$ *but* $x_2 \neq y_2$. *When is* $\alpha < \beta$*? Generalize.*

(d) *Suppose that* $j < k$ *and that* $x_0 = y_0$, $x_1 = y_1$, …, $x_j = y_j$. *When is* $\alpha < \beta$*?*

(e) *Evaluate the rational number* $\langle 0; 3, 1, 5, 99, 1\rangle$. *Does this affect your answer to part (d)?*

(a) One key fact is that if $\langle z_1; z_2, \dots, z_m\rangle$ is a continued fraction with *positive integer* entries z_i, then $1/\langle z_1; z_2, \dots, z_m\rangle$ is between 0 and 1. The short version is: $\langle z_1; z_2, \dots, z_m\rangle$ is z_1 plus something positive, and $z_1 \geq 1$. (A more rigorous version would establish this fact by induction on the length of the continued fraction.) In particular, $\alpha = x_0 + 1/\langle x_1; x_2, \dots, x_j\rangle$ is between x_0 and $x_0 + 1$, while $\beta = y_0 + 1/\langle y_1; y_2, \dots, y_k\rangle$ is between y_0 and $y_0 + 1$. Therefore, if $x_0 < y_0$, then $x_0 \leq y_0 - 1$ and so $\alpha < \beta$; on the other hand, if $x_0 > y_0$, then $x_0 \geq y_0 + 1$ and so $\alpha > \beta$.

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(b) Assuming that $x_1 \neq y_1$, by part (a) we have $\langle x_1; x_2, \dots, x_j\rangle < \langle y_1; y_2, \dots, y_k\rangle$ if and only if $x_1 < y_1$. Since $\alpha = x_0 + 1/\langle x_1; x_2, \dots, x_j\rangle$ and $\beta = y_0 + 1/\langle y_1; y_2, \dots, y_k\rangle = x_0 + 1/\langle y_1; y_2, \dots, y_k\rangle$, we conclude that $\langle x_1; x_2, \dots, x_j\rangle < \langle y_1; y_2, \dots, y_k\rangle$ if and only if $\alpha > \beta$ (since the reciprocal reverses the inequality).

(c) Assuming that $x_2 \neq y_2$, by part (a) we have $\langle x_2; x_3, \dots, x_j\rangle < \langle y_2; y_3, \dots, y_k\rangle$ if and only if $x_2 < y_2$. Since

$$\alpha = x_0 + \cfrac{1}{x_1 + \cfrac{1}{\langle x_2; x_3, \dots, x_j\rangle}} \quad \text{and} \quad \beta = y_0 + \cfrac{1}{y_1 + \cfrac{1}{\langle y_2; y_3, \dots, y_k\rangle}} = x_0 + \cfrac{1}{x_1 + \cfrac{1}{\langle y_2; y_3, \dots, y_k\rangle}},$$

we conclude that $\alpha < \beta$ if and only if $x_2 < y_2$ (since the double reciprocal ends up preserving the original inequality direction).

In general: if $x_0 = y_0, \dots, x_{\ell-1} = y_{\ell-1}$, but $x_\ell < y_\ell$, then $\alpha < \beta$ if ℓ is even, while $\alpha > \beta$ if ℓ is odd. (Remember that this is only for continued fractions with integer entries.) The sound bite reason is that an even number of reciprocals preserves inequalities, while an odd number of reciprocals reverses inequalities.

This funny ordering on expressions of the form $\langle x_0; x_1, \dots, x_j\rangle$ is called the *alternating lexicographic ordering*, or *alt-lex ordering* for short. A pure lexicographic ordering is how you order words in a dictionary (lexicon): you sort by first letter in ascending order, then break ties by sorting by second letter in ascending order, and so on. In alt-lex ordering, we sort by "first letter" (in this case, the value of x_0) in ascending order as usual; but then we break ties by sorting by "second letter" (x_1) in *descending* order, then by x_2 in ascending order, by x_3 in descending order, etc. This alt-lex ordering would correspond to a dictionary where A came before B, while AT came before(!) AS, while ARE came before ARF, while ARES came before(!) AREA, and so on.

(d) The answer is that $\alpha < \beta$ when j is even, while $\alpha > \beta$ when j is odd. One way to see this is to compare $\alpha = \langle x_0; x_1, \dots, x_j\rangle$ to

$$\beta = \langle y_0; y_1, \dots, y_k\rangle = \big\langle x_0; x_1, \dots, x_{j-1}, \langle x_j; y_{j+1}, \dots, y_k\rangle\big\rangle$$

and use the above arguments to compare x_j with $\langle x_j; y_{j+1}, \dots, y_k\rangle$. Another (slightly fishy) way to see this is to write $\alpha = \langle x_0; x_1, \dots, x_j\rangle = \langle x_0; x_1, \dots, x_j, \infty\rangle$ (check this!) and then use part (c) directly.

(e) By the second identity in #1(d), we see that

$$\langle 0; 3, 1, 5, 99, 1\rangle = \left\langle 0; 3, 1, 5, 99 + \frac{1}{1}\right\rangle = \langle 0; 3, 1, 5, 100\rangle = \frac{601}{2304}$$

by #1(c). But this makes us realize that there is one exception to the general statement in part (c): it is possible for $x_0 = y_0, \dots, x_{\ell-1} = y_{\ell-1}$, but $x_\ell < y_\ell$, and yet $\alpha = \beta$: namely, the two continued fractions

$$\langle x_0; x_1, \dots, x_{j-1}, x_j\rangle \quad \text{and} \quad \langle x_0; x_1, \dots, x_{j-1}, x_j - 1, 1\rangle$$

are equal. It's not hard to show that this is the only such exception; otherwise, distinct continued fractions (with integer entries) are indeed unequal.