**Math 539**
**Comments on Homework #1**

I(b). There are several ways to express what was wrong with the "proof". One way is to say that in $\sum_{n<N} O(n)$, the $O$-constant depended on $n$, and so the manipulation $\sum_{n<N} O(n) = O\big(\sum_{n<N} n\big)$ is invalid. Another way is to say that all these $O$-constants are the same, but they depend on the bounded range $[0, N]$ referenced in part (a), and so $O(N^2)$ is really the (meaningless) $O_N(N^2)$.

III(c). Although this could be done by hand in various ways, it follows quickly from part (b). For example, when $b < 1$ we certainly have $\log^b x \ll \varepsilon \log x$ for any $\varepsilon > 0$; exponentiating both sides—valid by part (b), since exponential functions are convex—immediately yields $\exp(\log^b x) \ll x^\varepsilon$. Similarly, starting from $A \log y \ll y^b$ for any $A, b > 0$, we substitute $y = \log x$ (valid by Reality Check V) to get $A \log\log x \ll (\log x)^b$, whence exponentiating again yields the desired $\log^A x \ll \exp(\log^b x)$.

Also in this question, some solutions came across the following useful observation: if $f(x) \ll g(x)$ for $x > x_0$, then it often follows that $f(x) \ll g(x)$ for all $x$. For example, if $f$ and $g$ are continuous on $[1, \infty)$ and $g(x) > 0$ there, and if we can prove that $f(x) \ll g(x)$ for $x > x_0$, then automatically $f(x) \ll g(x)$ on $[1, x_0]$, since the continuous function $f/g$ is bounded there. This implies that $f(x) \ll g(x)$ on all of $[1, \infty)$, with a $\ll$-constant that is the maximum of the two $\ll$-constants for $[1, x_0]$ and $(x_0, \infty)$. (The incantation to recite along with such an argument is "by increasing the implicit constant if necessary".)

V. Let $Q_k(x)$ denote the number of squarefree multiples of $k$ up to $x$, and let $\bar{Q}_k(x) = Q(x) - Q_k(x)$ denote the number of squarefree integers up to $x$ that are not multiples of $k$. Since the number of multiples of $k$ up to $x$ is $x/k + O(1)$ while the number of nonmultiples of $k$ is $x - x/k + O(1)$, the question is asking to compare

$$\lim_{x\to\infty} \frac{Q_k(x)}{x/k + O(1)} \quad \text{to} \quad \lim_{x\to\infty} \frac{\bar{Q}_k(x)}{x - x/k + O(1)}.$$

Noting, however, that $a/b < c/d$ if and only if $a/b < (a+c)/(b+d)$, it suffices to compare

$$\lim_{x\to\infty} \frac{Q_k(x)}{x/k + O(1)} \quad \text{to} \quad \lim_{x\to\infty} \frac{Q(x)}{x + O(1)} = \frac{6}{\pi^2}.$$

It also helps to see that $n$ is a squarefree multiple of $k$ if and only if $n = km$, where $m$ is squarefree and relatively prime to $k$. Therefore $Q_k(x) = R(x/m)$ where $R_k(y) = \sum_{n \le y} r_k(n)$ and $r_k(n)$ is the indicator function of squarefree integers that are relatively prime to $k$.

Determining the asymptotics of $R_k(y)$ is quite an instructive example. Since $\mu^2$, the indicator function of squarefree integers, decomposes as $1 * g$, where $g(n) =$

$\mu(m)$ if $n = m^2$ and 0 otherwise, we might try a similar decomposition of $r_k$. One possibility is to write $r_k = 1_k * g_k$, where

$$1_k(n) = \begin{cases} 1, & \text{if } (n,k) = 1, \\ 0, & \text{if } (n,k) > 1 \end{cases} \quad \text{and} \quad g_k(n) = \begin{cases} \mu(m), & \text{if } n = m^2 \text{ and } (m,k) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

(Concocting and verifying such convulation identities for multiplicative functions is an *invaluable* skill in multiplicative number theory. As mentioned in class, one of the best ways is to work out the effect on prime powers by hand.) From here we proceed as in the "naive" version of Dirichlet's hyperbola method, writing

$$R(y) = \sum_{n \le y} 1_k * g_k(n) = \sum_{c \le y} g_k(c) \bigg( \sum_{d \le y/c} 1_k(d) \bigg).$$

Asymptotics for the inner sum must be worked out (another instructive exercise—write $1_k(n) = e((n,k))$ and use the fact that $e = 1 * \mu$), and the resulting main-term sum over $c$ is easily seem to converge by the Claim from the lecture on Wednesday, September 14.

Another possibility is to write $r_k = 1 * h_k$ where $h_k$ is the multiplicative function defined on prime powers by

$$h_k(p^r) = \begin{cases} -1, & \text{if either } r = 2 \text{ and } p \nmid k \text{ or else } r = 1 \text{ and } p \mid k, \\ 0, & \text{otherwise.} \end{cases}$$

(Again, verify that this convolution is correct!) Then $R(y) = \sum_{c \le y} h_k(c) \lfloor y/c \rfloor$, so we didn't have to work out asymptotics for the inner sum as in the first possibility; however, deducing that this sum over $c$ converges is slightly less straightforward (although problem III on Homework #2 suffices).

VI(b). A clean way to see that $\tau(n) \ll n^\varepsilon$ cannot be uniform in $\varepsilon$ is as follows: if $\tau(n) \le Cn^\varepsilon$ for every $\varepsilon > 0$, with C independent of $\varepsilon$, then

$$\tau(n) = \lim_{\varepsilon \to 0+} \tau(n) \le \lim_{\varepsilon \to 0+} Cn^\varepsilon = C.$$

But $\tau(n)$ is not a bounded function (consider $\tau(p^r) = r + 1$ for example), so this is a contradiction.

VII. A few comments. First, when we have a sum like $\sum_{m \le M} m/p^m$, or even $\sum_{m \le M} 1/p^m$, one can try to derive an exact formula for the sum (not too bad for the second sum, rather annoying for the first sum). But if we're after an asymptotic formula, it's often easier to write the sum as (for example)

$$\sum_{m \le M} \frac{m}{p^m} = \sum_{m=1}^{\infty} \frac{m}{p^m} - \sum_{m > M} \frac{m}{p^m}.$$

The first term is a special value of the power series $\sum_{m=1}^{\infty} mz^m$, whose closed form is cleaner than any of its partial sums. The second term can be estimated as an error term. One useful observation is that if $f(x)$ is any polynomial, then

$$\sum_{m=0}^{\infty} \frac{f(m)}{p^m} \ll_f 1$$

uniformly in $p$. Why? Because this is the power series $\sum f(m)z^m$ evaluated at $1/p$. Any power series whose coefficients are the values of a polynomial has radius of convergence 1 (use the Ratio Test for example), thus defines an analytic function on the open unit disk $\{z \in \mathbb{C}\colon |z| < 1\}$. In particular, it is continuous on the compact smaller disk $\{z \in \mathbb{C}\colon |z| \le \frac{1}{2}\}$, and hence uniformly bounded on that set, which includes all numbers of the form $z = 1/p$. The dependence of the implicit constant on the polynomial $f$ can be attended to on a case-by-case basis.

Some solutions had final formulas for the average value that included terms like $p^{-\lfloor \log_p x \rfloor}$. Average values (as opposed to average orders) shouldn't have any variables $x$ or $n$ in them in general; for this particular quantity, we see that $p^{-\lfloor \log_p x \rfloor} \le p^{\log_p x - 1} = p/x$, so can easily be subsumed into the other error terms. As an aside, let's be reminded that the statement "$f(n)$ has average order $g(n)$" means that

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} f(n) \sim \lim_{x\to\infty} \frac{1}{x} \sum_{n \le x} g(n), \tag{Yes}$$

not

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} f(n) \sim g(x) \tag{No}$$

or

$$f(n) \sim g(n). \tag{No'}$$

For example, if $f(n) = n$, then $f(n)$ has average order $n$ according to (Yes), but $\frac{1}{x}\sum_{n\le x} f(n) = \frac{x}{2} + O(1)$, so (No) would incorrectly report that the average order of $f(n)$ is $\frac{n}{2}$. As another example, if $f(n) = 1 + (-1)^n$ (alternating values of 0 and 2), then the average order of $f$ according to (Yes) is 1, yet $f(n) \sim 1$ is false, exposing (No').

Technically speaking, "the" average order of an arithmetic function $f(n)$ is not well-defined, since if we say that the average order is $g(n)$, then any function $h(n) \sim g(n)$ also satisfies the asymptotic relation (Yes). Does $\tau(n)$ have average order $\log n$ or $\log n + 2\gamma - 1$? Technically both are true, although notice that the second one is *less* true than saying it has average order $\log n + 2\gamma$ (why?). In general, we either say "average order" with as simple a function as possible, or else we write the asymptotic relation (Yes) out explicitly, showing the quality of the error term.

VIII. The first two equalities in part (a) are correct, but from there it should go:

$$\sum_{n\le x} \sum_{p_1|n} \sum_{p_2|n} 1 = \sum_{p_1 \le x} \sum_{\substack{p_2\le x \\ p_2 \ne p_1}} \sum_{\substack{n \le x \\ p_1p_2|n}} 1 + \sum_{p\le x} \sum_{\substack{n\le x \\ p|n}} 1,$$

taking into account the occasions where $p_1 = p_2$. The condition $p_2 \ne p_1$ is annoying, but we can now adjust once again:

$$\sum_{p_1 \le x} \sum_{\substack{p_2\le x \\ p_2 \ne p_1}} \sum_{\substack{n \le x \\ p_1p_2|n}} 1 + \sum_{p\le x} \sum_{\substack{n\le x \\ p|n}} 1 = \Bigg( \sum_{p_1 \le x} \sum_{p_2 \le x} \sum_{\substack{n\le x \\ p_1p_2|n}} 1 - \sum_{p\le x} \sum_{\substack{n \le x \\ p^2|n}} 1 \Bigg) + \sum_{p\le x} \sum_{\substack{n\le x \\ p|n}} 1.$$

From here it seems most natural to write the first sum as

$$= \sum_{p_1 \le x} \sum_{p_2 \le x} \left\lfloor \frac{x}{p_1 p_2} \right\rfloor = \sum_{p_1 \le x} \sum_{p_2 \le x} \left( \frac{x}{p_1 p_2} + O(1) \right);$$

however, notice that there are now $\pi(x)^2 \sim x^2/\log^2 x$ error terms, far larger than the expected main term (even the trivial upper bound $\omega(n) \le \log_2 n$ leads to $\sum_{n \le x} \omega^2(n) \ll x \log^2 x$, so an error or $\pi(x)^2$ is going to be unacceptable). Notice though that we can write this term as

$$= \sum_{\substack{p_1, p_2 \\ p_1 p_2 \le x}} \left\lfloor \frac{x}{p_1 p_2} \right\rfloor = \sum_{\substack{p_1, p_2 \\ p_1 p_2 \le x}} \left( \frac{x}{p_1 p_2} + O(1) \right),$$

since the summand equals 0 when $p_1 p_2 > x$; now the number of error terms is much smaller (in truth $x \log\log x/\log x$, although a simpler estimate leads to $x \log\log x$ which is still good enough). The main term can be dealt with using Dirichlet's hyperbola method. (Think of it, as you like, as the summatory function of $g * g$, where $g(n)$ is the function that equals $1/n$ if $n$ is prime and 0 otherwise.)