Math 539
Comments on Homework #2

I. This is quite a typical problem for these sorts of methods. During the process of using the hyperbola method, we encounter the necessity of understanding the sums $\sum_{n \leq x}(\log n)/n$ and $\sum_{n \leq x} \tau(n)/n$, each of which is a mini-problem in itself, and that's typical. For the latter sum, I saw successful evaluations both by writing the summand as $T^{-1}\tau = T^{-1}1 * T^{-1}$ and using the hyperbola method again, and also by computing

$$\int_{1-}^{x} \frac{1}{t} \, d\left(\sum_{n \leq t} \tau(n) \right)$$

and summing by parts.

In either case, doing the hyperbola method on the original $\sum_{n \leq x}(\tau * 1)(n)$ yields the mess

$$\frac{x}{2}\left(\log^2 y + 2\log x \log \frac{x}{y} + \log^2 \frac{x}{y} \right) + x\left(2\gamma \log y + (2\gamma - 1)\log \frac{x}{y} + \gamma \log x - \log y \right)$$

$$+ x(\text{some constants}) + O\left(\frac{x}{\sqrt{y}} \log y + y \log x + y \log \frac{x}{y} + \frac{x}{\sqrt{y}} + y \log y + \frac{x}{\sqrt{y}} \right).$$

Note that the main terms actually don't depend on y, as the first two can be simplified to

$$\frac{x}{2}\left(\log y + \log \frac{x}{y} \right)^2 + x(3\gamma - 1)\log x = \frac{x \log^2 x}{2} + x(3\gamma - 1)\log x,$$

and this too is typical for the hyperbola method (the main terms shouldn't depend on the order in which we count things!). Finally, most of the terms in the error are dominated by a couple important ones; in fact, the entire error is $O\left(\log y(x/\sqrt{y} + y) \right)$. We said in class that the quickest way to minimize this expression is not to use calculus (which doesn't even work out in this case), but to simply set the increasing and decreasing terms equal to each other to choose the parameter y. This is typical as well; in this particular case, solving $x/\sqrt{y} = y$ yields $y = x^{2/3}$, which minimizes the size of the error term as $O(x^{2/3} \log x)$. Note in particular that choosing $y = \sqrt{x}$ is not always best.

V(a). The fact that $\int_{1-}^{x} f(t) \, d\lfloor t \rfloor = \int_{1-}^{x} f(t) \, dt - \int_{1-}^{x} f(t) \, dB_1(\{t\})$ is immediate from Reality Check IV, upon noting that $\lfloor t \rfloor = t - B_1(\{t\}) - \frac{1}{2}$.

V(b). Plugging $f(t) = 1/t$ directly into part (a) yields the answer we want, except that we have some unspecified constant C rather than the desired γ. However, just subtracting the known asymptotic formula $\sum_{n \leq x} 1/n = \log x + \gamma + O(1/x)$ and taking the limit immediately shows that $C = \gamma$ without looking at the specific expression for C.

VI(b). Many of you computed the values for $\tau_{1/2}(p^k)$ recursively by hand up to $k = 4$, which is completely sufficient. I did want to mention, however, a way of calculating a formula for $\tau_{1/2}(p^k)$ all in one go. Given an arithmetic function f and a prime p, the *Bell series* $B_{f,p}(x)$ for f is the power series $\sum_{k=0}^{\infty} f(p^k)x^k$. While this is defined for any arithmetic function f, it is most useful when f is multiplicative, and especially when $f(p^k)$ depends only on k and not p. The important fact about Bell series, which you should verify, is that $B_{f*g,p}(x) = B_{f,p}(x)B_{g,p}(x)$ for any arithmetic functions f and g.

So in this case, we want $\tau_{1/2} * \tau_{1/2} = \tau_1 = 1$. The Bell series for $f = 1$ is simply $B_{1,p}(x) = \sum_{k=0}^{\infty} 1x^k = 1/(1-x)$ independent of p. Therefore $B_{\tau_{1/2},p}(x)$ should be a function that squares to $1/(1-x)$, namely $\pm 1/\sqrt{1-x}$. (Some of you noticed that there were two possibilities, and I agree that choosing the one for which $\tau_{1/2}(1) = 1$ is more natural.) Therefore from the binomial theorem for $(1-x)^{-1/2}$, we see that

$$B_{\tau_{1/2},p}(x) = \sum_{k=0}^{\infty} \binom{-1/2}{k}(-x)^k = \sum_{k=0}^{\infty} \binom{2k}{k}\left(\frac{x}{4}\right)^k,$$

where we used $\binom{-1/2}{k} = (-\frac{1}{2})(-\frac{3}{2})\ldots(-\frac{2k-1}{2})$ to obtain the final equality. This yields the formula $\tau_{1/2}(p^k) = \binom{2k}{k}/4^k$.

In general, a similar argument yields $\tau_\alpha(p^k) = (-1)^k\binom{-\alpha}{k} = \binom{\alpha+k-1}{k}$ for any *complex* number α even—a fact I didn't know until after I assigned this problem!

VII. In a sense, this problem and problem V from the first homework are the most important problems so far, since they give practice in translating an imprecise idea (that might come up while trying to solve some more complicated problem) into a rigorous statement. In this case, the quantity to be considered is

$$\frac{\sum\limits_{n\leq x} w(n)}{\sum\limits_{n\leq x} 1} - \frac{\sum\limits_{\substack{n\leq x \\ (n,q)=1}} w(n)}{\sum\limits_{\substack{n\leq x \\ (n,q)=1}} 1};$$

note in particular the denominator of the second fraction (the average value is the sum of the values divided by the number of terms, which is not always simply x).

Also, a few students ran into trouble when switching the order of summation in the double sum

$$\sum_{\substack{n\leq x \\ (n,q)=1}} \sum_{p|n} 1 = \sum_{\substack{p\leq x \\ p\nmid q}} \sum_{\substack{m\leq x/p \\ (m,q)=1}} 1;$$

note that the $(m,q) = 1$ condition in the last sum is still necessary. Although the asymptotic formula

$$\sum_{\substack{m\leq y \\ (m,q)=1}} 1 = \frac{\phi(q)}{q}x + O(q)$$

is easily derived by considering blocks of q consecutive integers, a way to get a better error term is to write

$$\sum_{\substack{m \leq y \\ (m,q)=1}} 1 = \sum_{m \leq y} e((m,q)) = \sum_{m \leq y} \sum_{d|(m,q)} \mu(d)$$

and switch the order of summation; you should verify that this gives an error term of $O(2^{\omega(q)})$, which we now know (from the upper bound $\omega(q) \ll \log q / \log \log q$) is $O_\varepsilon(q^\varepsilon)$ for any $\varepsilon > 0$.

VIII. Note that we should demand $0 < \alpha < 2$ instead of $0 < \alpha \leq 2$. Apparently the function $\sin x$ in part (a) was so specific that all but one person did the problem by hand (correctly) instead of the way I had intended, through integration by parts:

$$\int_0^\infty \frac{\sin x}{x^\alpha} \, dx = \int_0^1 \frac{\sin x}{x^\alpha} \, dx + \int_1^\infty \frac{\sin x}{x^\alpha} \, dx$$

$$= \int_0^1 \frac{\sin x}{x^\alpha} \, dx - \frac{\cos x}{x^\alpha}\Big|_1^\infty - \alpha \int_1^\infty \frac{\cos x}{x^{\alpha+1}} \, dx$$

$$\ll_\alpha \int_0^1 \frac{dx}{x^\alpha} + 1 + \int_1^\infty \frac{dx}{x^{\alpha+1}} < \infty.$$

The point of integrating by parts to show that the integral converges at ∞ is to use the fact that sin oscillates so much that its integral (namely $-\cos$) is bounded.

 This is an *exact* analogue with the summation-by-parts solution to part (b): the function χ oscillates so much that its summatory function $S(x) = \sum_{n \leq x} \chi(n)$ is bounded, and hence

$$\int_{1-}^\infty \frac{1}{x^\beta} \, dS(x) = \frac{S(x)}{x^\beta}\Big|_{1-}^\infty + \beta \int_1^\infty \frac{S(x)}{x^{\beta+1}} \, dx \ll_{\chi,\beta} \int_1^\infty \frac{dx}{x^{\beta+1}} < \infty.$$

In fact, this is precisely the proof of "Dirichlet's Test" which one of you quoted (and which I had never heard of—thanks for the tip!).

IX. The key to doing this problem (in the way I envisioned, at least) is to find and exploit the identity

$$\frac{n}{\phi(n)} = \sum_{d|n} \frac{\mu^2(d)}{\phi(d)}.$$

Two hints to look for an identity of the form $n/\phi(n) = \sum_{d|n} f(d)$ might be: (1) the function $n/\phi(n)$ is roughly constant, and so the function $f(d)$ might be quite small, which helps with evaluating sums; (2) $n/\phi(n)$ depends only upon the primes dividing n, not their multiplicities, and so the corresponding $f(d)$ will be supported on squarefree numbers.

 Once we decide to look for such an identity, it's best to look at one prime at a time, since we're dealing with multiplicative functions. We can use a recursive method to guess or establish a pattern, or we can use the Bell series defined above for a more systematic approach.

X(a). One nice way to see that $f = r * s$ is as follows. Define χ_k to be the character-istic function of kth powers. Then the identites $r = \chi_2 * \chi_3$ and $s * \chi_6 = e$ are more or less obvious from the definitions of the functions involved. Therefore it is equivalent to show that $f * \chi_6 = \chi_2 * \chi_3$. Moreover, all four functions in this last expression are multiplicative, and so it suffices to show equality on prime powers. This is easily done by hand or by considering Bell series as defined above.

X(c). If $PF(x)$ denotes the number of powerfull integers up to x, then you proved in this problem that

$$PF(x) = \frac{\zeta\left(\frac{3}{2}\right)}{\zeta(3)} x^{1/2} + \frac{\zeta\left(\frac{2}{3}\right)}{\zeta(2)} x^{1/3} + O\left(x^{1/5}\right). \tag{$*$}$$

I just want to point out how remarkable this is! I went ahead and calculated $PF(x)$ for all values of x up to one million, and I challenge anyone to be con-vinced from that data that $PF(x)$ is even asymptotic to *any* constant times $x^{1/2}$. Doing a linear regression on the log-log plot gives the not-very-impressive guess $PF(x) \approx 1.562 x^{0.519}$.

Even if a guardian angel told us that $PF(x) \sim Ax^{1/2} + Bx^{1/3}$, a regression to find the best-fitting constants in this formula gives $(A, B) \approx (2.1800, -1.5616)$, which I doubt would immediately bring to mind the true constants

$$\left(\frac{\zeta\left(\frac{3}{2}\right)}{\zeta(3)}, \frac{\zeta\left(\frac{2}{3}\right)}{\zeta(2)} \right) \approx (2, 1733, -1.4880).$$

(You did know that $\zeta\left(\frac{2}{3}\right)$ was negative, right?) Yet when tested against the asymp-totic formula $(*)$, which you derived by hand in a couple of pages, the data reveals that the error term in $(*)$ is in fact bounded in absolute value by *five* in this range.

(In case anybody cares, there are 2,027 powerfull numbers up to 10^6; the 2,027th is of course one million itself, while the 2,026th one is the less well-known $998{,}784 = 2^7 3^3 17^2$.)