

**Math 539**  
**Comments on Homework #3**

I(a). What I was thinking, when I originally wrote the problem with the erroneous factor of  $\pi^2/6$ , was the bound on  $\sigma(n)$  for *squarefree*  $n$ , which turns out to be:

$$\mu^2(n)\sigma(n) < \frac{6}{\pi^2}e^\gamma n \log \log n + O(n).$$

In fact this is even easier to establish than the problem you were given: the key is that for squarefree  $n$ , we have

$$\frac{\sigma(n)}{n} = \prod_{p|n} \left(1 + \frac{1}{p}\right) = \frac{n}{\phi(n)} \prod_{p|n} \left(1 - \frac{1}{p^2}\right).$$

However, this does point something out: proving that  $\sigma(n)$  is sometimes as large as  $e^\gamma n \log \log n + O(n)$  might not be easy, because we'll never get such large values of  $\sigma(n)$  by restricting to squarefree  $n$ . Here we miss by a factor of  $1/\zeta(2)$ ; if we broaden our criterion but still restrict to integers  $n$  that are not divisible by a nontrivial  $k$ th power, then an equally straightforward argument yields the upper bound  $(1/\zeta(k))e^\gamma n \log \log n + O(n)$ .

So to show that the upper bound given in this problem is best possible, we do need to look at numbers that are quite non-squarefree—in fact, numbers that are divisible by arbitrarily high powers. Still, some reasonably compact and simply-defined numbers should do the trick. Two choices that can be used to finish this problem off (there are surely others) are to choose products of primes whose exponents:

- gradually decrease as  $p$  grows, such as  $\prod_{p \leq y} p^{\lfloor \log x / \log p \rfloor}$ ; or
- are large for small  $p$  and then abruptly shrink, such as  $\prod_{p \leq z} p^k \prod_{z < p \leq y} p$ , where  $k$  and  $z$  go to infinity but suitably slowly compared to  $y$ .

I(b). As you know, this problem boils down to finding the nonnegative integer  $k$  that maximizes the function  $f_{\varepsilon,p}(k) = (k+1)/p^{k\varepsilon}$ . One can certainly do this by looking at  $f'_{\varepsilon,p}(x)$ , finding the positive real number that maximizes the function, and then trying to decide which nearby integer to look at. However, don't forget the discrete version of the derivative, namely the difference! Determining where the expressions  $f_{\varepsilon,p}(k) - f_{\varepsilon,p}(k \pm 1)$  are positive immediately shows that the optimal value of  $k$  is  $k = \lfloor (p^\varepsilon - 1)^{-1} \rfloor$ .

As a side note: the primes between  $(3/2)^{1/\varepsilon}$  and  $2^{1/\varepsilon}$  played an important role (these are the ones for which  $k = 1$  is optimal), and this comprises most of the relevant primes, since the lower bound is the  $\alpha$ th power of the upper bound where  $\alpha = \log(3/2)/\log 2$ . Would it surprise you to know that I have the fact  $\alpha \approx 7/12$  stored in my head? (In fact the approximation is good to within 0.3%.) The reason I know this comes from music, of all places! An octave consists of two notes whose frequencies have a ratio of 2 : 1, whereas a perfect fifth consists of two notes whose frequencies have a ratio of 3 : 2. The entire foundation of our musical notation system, with twelve semitones per octave, is based on the mathematical fact that twelve perfect fifths is approximately the same as seven octaves; and this in turn is a consequence of the approximation  $\log(3/2)/\log 2 \approx 7/12$ .

II(c). The fact that  $\tau(n) = (\log n)^{\log 2 + o(1)}$  for almost all  $n$ , together with the fact that  $\tau(n)$  is about  $\log n$  on average, shows that the average value is being influenced by relatively few (density 0, in other words) abnormally large values of  $\tau$ . One of the problems on Homework #5 will explore this idea in greater detail.

VI. Most people got this problem essentially correct, and many shrewdly used the fact that

$$F'(0) = F(0) \cdot \frac{F'(0)}{F(0)} = F(0) \cdot \left. \frac{d(\log F(z))}{dz} \right|_{z=0}$$

to reduce the problem to calculating the derivative at 0 of the infinite series

$$\log F(z) = \sum_p \left( \log \left( 1 - \frac{1}{p} \right) + \log \left( 1 + \frac{1}{p-z} \right) \right). \quad (**)$$

Nobody got this problem 100% correct, however: it is a subtle point to justify that  $\frac{d}{dz}(\log F(z))$  can be calculated by taking the term-by-term derivative of the right-hand side.

The main theorem in this regard is: *If  $F(x) = \sum_{k=1}^{\infty} g_k(x)$  converges uniformly on a domain  $D$ , and if each  $g_k(x)$  is differentiable on  $D$  and if  $\sum_{k=1}^{\infty} g'_k(x)$  converges to some function uniformly on  $D$ , then that limit function is the derivative of the original function, that is,  $F'(x) = \sum_{k=1}^{\infty} g'_k(x)$ . (There's a similar theorem for the situation  $F(x) = \int_0^{\infty} f(x, y) dy$ .)*

To justify differentiating under the summation sign, therefore, we need that the two series both converge uniformly. Here's an example, communicated to me by Glen Pugh, that shows that such an assumption really is necessary. Set

$$g_k(x) = \frac{x}{1 + (k-1)^2 x^2} - \frac{x}{1 + k^2 x^2}.$$

Then we have the telescoping sum

$$\sum_{k=1}^n g_k(x) = x - \frac{x}{1 + n^2 x^2},$$

and so

$$F(x) = \sum_{k=1}^{\infty} g_k(x) = \lim_{n \rightarrow \infty} \left( x - \frac{x}{1 + n^2 x^2} \right) = x$$

for all real numbers  $x$ . Furthermore, the convergence of the infinite sum defining  $F$  is uniform on every set of the form  $\mathbb{R} \setminus (-\varepsilon, \varepsilon)$ , as the difference between the value of the partial sum,  $x - x/(1 + n^2 x^2)$ , and the limit value  $x$  is uniformly bounded by  $\varepsilon/n^2$  on such a set. However, the convergence of the sum defining  $F$  is not uniform on  $\mathbb{R}$ , in particular not uniform near  $x = 0$ . (Note that the point  $x = 0$  itself is not a problem at all, as  $g_k(0) = 0$  for all  $k$ .)

However, we have

$$g'_k(x) = \frac{1 - (k-1)^2 x^2}{((k-1)^2 x^2 + 1)^2} - \frac{1 - k^2 x^2}{(k^2 x^2 + 1)^2},$$

and hence we have another telescoping sum

$$\sum_{k=1}^n g'_k(x) = 1 - \frac{1 - n^2 x^2}{(n^2 x^2 + 1)^2}.$$

Again, these partial sums converge uniformly on sets of the form  $\mathbb{R} \setminus (-\varepsilon, \varepsilon)$ , and so the theorem above implies that

$$F'(x) = \sum_{k=1}^{\infty} g'_k(x) = \lim_{n \rightarrow \infty} \left( 1 - \frac{1 - n^2 x^2}{(n^2 x^2 + 1)^2} \right) = 1$$

on any such set, in particular for any  $x \neq 0$ . However,

$$\sum_{k=1}^{\infty} g'_k(0) = \lim_{n \rightarrow \infty} \left( 1 - \frac{1 - n^2 0^2}{(n^2 0^2 + 1)^2} \right) = \lim_{n \rightarrow \infty} 0 = 0 \neq 1 = F'(0).$$

(Alternatively,  $\sum g'_k(0) = \sum 0 = 0$ .) One might complain that differentiating under the summation sign is *almost* right ... but in this very problem we're concerned with  $F'(0)$ ! So we really do need some information about uniform convergence.

Back to the problem at hand: it is not hard to show that on the unit disk, for example, both the right-hand side of (\*\*\*) and its term-by-term derivative

$$\sum_p \frac{1}{(p-z)(p-z+1)}$$

converge uniformly: both tails can be shown to be  $O(\sum_{p>y} 1/p^2)$  uniformly for  $|z| \leq 1$ . Therefore in this case, differentiation under the summation sign is permissible.

VIII. I note only that we can factor 57-digit numbers with ease (my office computer factored a dozen randomly-chosen ones and the longest running time was under 90 seconds); hence once we find the 56-digit candidate  $n$  for a solution to  $\phi(6n + 1) < \phi(6n)$ , we can bypass the estimates analogous to those in my paper and simply compute  $\phi(6n + 1)$  and  $\phi(6n)$  exactly.

IX(c). The last question is intentionally misleading. Let's remind ourselves again: a function  $f(n)$  has average order  $g(n)$  if  $\sum_{n \leq x} f(n)$  and  $\sum_{n \leq x} g(n)$  grow at the same rate. So to see how good a fit a potential function  $g(n)$  is to be the average order of  $f(n)$ , we need to calculate  $g(n)$ . Considering functions of the form  $\log n + C$  as candidates for the average order of  $\tau(n)$ , we calculate (using integrals as upper and lower bounds):

$$\sum_{n \leq x} (\log n + C) = x \log x + (C - 1)x + O(\log x)$$

and

$$\sum_{n \leq x} (\log n + C) \left(1 - \frac{n}{x}\right) = \frac{1}{2}x \log x + \left(\frac{C}{2} - \frac{3}{4}\right)x + O(\log x).$$

So although every function of the form  $\log n + C$  is a reasonably good choice for the average order of  $\tau(n)$ , the function  $\log n + 2\gamma$  (not  $\log n + 2\gamma - 1$  and not  $\log n + 2\gamma - \frac{3}{2}$ ) is the best choice of them all, since it matches the secondary main terms as well.