Math 539
Comments on Homework #4

GENERAL COMMENT: If we bound part of a summand or integrand, we need to add absolute values on the other parts. For example, if $A(n) \ll 1$, then we have $\sum_{n=1}^{\infty} A(n)f(n) \ll \sum_{n=1}^{\infty} |f(n)|$, but not necessarily $\sum_{n=1}^{\infty} A(n)f(n) \ll \sum_{n=1}^{\infty} f(n)$; if $A(t) \ll A(x)$ for all $t \le x$, then we have $\int_0^x A(t)f(t)\,dt \ll A(x)\int_0^x |f(t)|\,dt$, but not necessarily $\int_0^x A(t)f(t)\,dt \ll A(x)\int_0^x f(t)\,dt$; if $0 \le a_n \le b_n$, then we have $\sum_{n\le x} a_n n^{-s} \ll \sum_{n\le x} b_n n^{-\sigma}$, but not necessarily $\sum_{n\le x} a_n n^{-s} \ll \sum_{n\le x} b_n n^{-s}$; and so on. Moreover, even if $A(x) \le B(x)$ are both nonnegative functions, the reasonable-seeming Riemann-Stieltjes inequality $\int_a^b A(x)\,dG(x) \le \int_a^b B(x)\,dG(x)$ is false in general! One would need $G(x)$ to be increasing, for example. The moral of the story: think twice, estimate once.

I. I just wanted to point out that this problem is another instance of the analytic tenet "you can't have half a pole". If the Dirichlet series had a pole somewhere on that vertical line, then near the pole ρ the function would have to blow up in size like $(s-\rho)^{-k}$ for some positive integer k; $k = \frac{1}{2}$ is not a possibility. (Note that for non-pole singularities such behavior is quite possible—the square root and logarithm functions near 0 are examples, as are essential singularities like $e^{-1/z}$. But not poles.)

II. At first I too had a hard time seeing why $\int_1^{\infty} \{x\}/x^{\sigma+1}\,dx < \frac{1}{2\sigma}$ (getting $\frac{1}{\sigma}$ as the upper bound is easy from $\{x\} \le 1$). It's not just that $\{x\}$ is $\frac{1}{2}$ on average; we need to somehow use that $\{x\}$ is less than $\frac{1}{2}$ first and greater than $\frac{1}{2}$ next, and since it's multiplied by a decreasing function, the less-than-$\frac{1}{2}$ parts should dominate. One way to carry this out is to write, for each unit interval $[n, n+1]$,

$$\begin{aligned}
\int_n^{n+1} \frac{\{x\} - 1/2}{x^{\sigma+1}}\,dx &= \int_0^1 \frac{y-1/2}{(n+y)^{\sigma+1}}\,dy \\
&= \int_0^{1/2} \frac{y-1/2}{(n+y)^{\sigma+1}}\,dy + \int_{1/2}^1 \frac{y-1/2}{(n+y)^{\sigma+1}}\,dy \\
&= \int_0^{1/2} \frac{y-1/2}{(n+y)^{\sigma+1}}\,dy + \int_0^{1/2} \frac{1-z-1/2}{(n+1-z)^{\sigma+1}}\,dz \\
&= \int_0^{1/2} (y-1/2)\left(\frac{1}{(n+y)^{\sigma+1}} - \frac{1}{(n+1-y)^{\sigma+1}}\right) dy \le 0.
\end{aligned}$$

Another, slicker way is to notice that $f(y) = \int_1^y \{t\}\,dt \le (y-1)/2$ for all $y \ge 1$ (consider each unit interval separately), so we can integrate by parts:

$$\int_1^{\infty} \frac{\{x\}}{x^{\sigma+1}}\,dx = \frac{f(x)}{x^{\sigma+1}}\bigg|_1^{\infty} + (\sigma+1)\int_1^{\infty} \frac{f(x)}{x^{\sigma+2}}\,dx;$$

the boundary terms are both 0, and the new integrand is nonnegative and

$$\begin{aligned}(\sigma+1)\int_1^\infty \frac{f(x)}{x^{\sigma+2}}\,dx &\le (\sigma+1)\int_1^\infty \frac{(x-1)/2}{x^{\sigma+2}}\,dx \\ &= \frac{\sigma+1}{2}\left(-\frac{1}{\sigma x^\sigma}+\frac{1}{(\sigma+1)x^{\sigma+1}}\right)\bigg|_1^\infty \\ &= \frac{\sigma+1}{2}\left(\frac{1}{\sigma}-\frac{1}{\sigma+1}\right)=\frac{1}{2\sigma}.\end{aligned}$$

The moral of this story: to take advantage of cancellation, put the sum/integral that has the cancellation in the innermost position. Summation/integration by parts often accomplishes this.

V. It might seem that the most natural way to do this problem is to prove that for all $\sigma \le 0$, the series $A(\sigma)$ converges if and only if $R(x) \ll x^\sigma$, but this equivalence is false. As it happens, the convergence of $A(\sigma)$ does imply that $R(x) = o(x^\sigma)$ even; however, $R(x) \ll x^\alpha$ implies only that $A(\alpha+\varepsilon)$ converges for every $\varepsilon > 0$, but not necessarily that $A(\alpha)$ itself converges. Fortunately, the abscissa of convergence and the quantity described in the question are both infema, and so we only have to show that

$$A(\sigma+\varepsilon) \text{ converges for every } \varepsilon > 0 \quad\Longleftrightarrow\quad R(x) \ll x^{\sigma+\varepsilon} \text{ for every } \varepsilon > 0.$$

(Technical note: according to the definition of infemum, we don't need the above statements for every $\varepsilon > 0$, but only for a sequence of ε tending to 0 from above. Fortunately, however, if $A(\sigma_0)$ converges then $A(\sigma)$ converges for every $\sigma > \sigma_0$, and similarly for upper bounds for $R(x)$; therefore we don't have to speak in terms of sequences of ε—either it works for all $\varepsilon > 0$ or it stops working below some threshhold ε.)

VI. Many of you experimented with the tables of values of $\sigma(n)^2$ and $\sigma(n^2)$ and related multiplicative functions on prime powers, finding by hand convolution identities that eventually reduced the problem to recognizing familiar Euler products. This is a perfectly reasonable approach. I did want to remind you that there is a more "mindless" approach, namely simply to compute the Bell series for these functions. For example, looking at $\sigma(n)^2$:

$$\begin{aligned}B_p(x) = \sum_{k=0}^\infty \sigma(p^k)^2 x^k = \sum_{k=0}^\infty \left(\frac{p^{k+1}-1}{p-1}\right)^2 x^k \\ &= \frac{1}{(p-1)^2}\sum_{k=0}^\infty \left(p^{2k+2}-2p^{k+1}+1\right)x^k \\ &= \frac{1}{(p-1)^2}\left(\frac{p^2}{1-p^2x}-\frac{2p}{1-px}+\frac{1}{1-x}\right) \\ &= \cdots = \frac{1+px}{(1-x)(1-px)(1-p^2x)}\end{aligned}$$

after a lot of algebra. Therefore the factor corresponding to p in the Euler product of $\sum_{n=1}^\infty \sigma(n)^2 n^{-s}$ is

$$B_p(p^{-s}) = \frac{1+p^{1-s}}{(1-p^{-s})(1-p^{1-s})(1-p^{2-s})} = \frac{1-p^{2-2s}}{(1-p^{-s})(1-p^{1-s})^2(1-p^{2-s})},$$

whence

$$\sum_{n=1}^\infty \sigma(n)^2 n^{-s} = \prod_p B_p(p^{-s}) = \prod_p \frac{1-p^{2-2s}}{(1-p^{-s})(1-p^{1-s})^2(1-p^{2-s})} = \frac{\zeta(s)\zeta(s-1)^2\zeta(s-2)}{\zeta(2s-2)}.$$

VII. Most of you gave the solution I had in mind, namely showing that $\eta(s)+\zeta(s+\alpha)$ had $\sigma_c = 1-\alpha$ and $\sigma_a = 1$. However, the argument that $\sigma_a = 1$ (for example) was flawed in some solutions. Showing that $\sum_{n=1}^\infty \big((-1)^{n-1}+n^{-\alpha}\big)n^{-s}$ converges absolutely for $\sigma > 1$ does not prove that $\sigma_a = 1$, but only that $\sigma_a \le 1$. In general, the mistake boils down to claiming that $\sigma_a(f+g) = \max\{\sigma_a(f), \sigma_a(g)\}$ rather than the accurate $\sigma_a(f+g) \le \max\{\sigma_a(f), \sigma_a(g)\}$. Note that the equality is provably true (I believe) if $\sigma_a(f) \ne \sigma_a(g)$, but certainly not in general: $f+g$ might be constant, for example.

VIII(b). Several of you went through a full inductive derivation parallel to the solution of part (a), but one can reduce this case directly to the result in part (a) as follows. Suppose that C and d are chosen so that $|f(n)| \le Cn^d$; then it can be checked that the function

$$g(n) = \frac{f(n)}{f(1)} n^{-d-\frac{\log(C/|f(1)|)}{\log 2}}$$

satisfies $g(1) = 1$ and $|g(n)| \le 1$ for all $n \in \mathbb{N}$. (This is just a specific way of increasing the d enough to overcome the unwanted constant C; increasing it enough so that the inequality holds for $n = 2$ is the worst case.) Part (a) shows that $g^{-1}(n)$ is of polynomial growth, and it follows easily that $f^{-1}(n)$ is then of polynomial growth, since the convolution inverse of $\lambda f(n)n^k$ is $\lambda^{-1}f^{-1}(n)n^k$.

VIII(c). The point here is that an arithmetic function $f(n)$ having polynomial growth is *equivalent* to the Dirichlet series $\sum_{n=1}^\infty f(n)n^{-s}$ converging in some half-plane (that is, having $\sigma_c < \infty$): polynomial growth implies convergence for large σ by comparison to the sum $\sum_{n=1}^\infty n^{-p}$, while convergence for some s implies the growth bound $f(n) \ll n^\sigma$ since the terms in a convergent series must tend to 0.