Math 539
Homework #2
due Wednesday, October 12, 2005 at 10 AM

Reality check problems. Not to write up; just ensure that you know how to do them.

I. Let $f(n)$ be a multiplicative function. Prove that for any fixed integer q, the function $f(qn)/f(n)$ is a multiplicative function of n.

II. Fill in the details of the proof of equation (3.5) in Lemma 3.13 of Bateman and Diamond (p. 48).

III. Bateman and Diamond, p. 68, #3.27

IV. If (F, G_1) and (F, G_2) are both compatible pairs of functions, prove that $(F, G_1 + G_2)$ is also a compatible pair and that

$$\int_a^b F(t)\,d(G_1+G_2)(t) = \int_a^b F(t)\,dG_1(t) + \int_a^b F(t)\,dG_2(t).$$

Conclude that for any compatible pair (F, G) and any constant c,

$$\int_a^b F(t)\,d(G(t)+c) = \int_a^b F(t)\,dG(t).$$

Homework problems. To write up and hand in.

I. Bateman and Diamond, p. 68, #3.28

II. Bateman and Diamond, p. 68, #3.29

III. Let f be a multiplicative function. We would like to have conditions under which we can conclude that both expressions

$$\sum_{n=1}^{\infty} f(n) \quad \text{and} \quad \prod_p \big(1 + f(p) + f(p^2) + \cdots\big) \quad \text{converge to equal values.} \tag{$*$}$$

We saw in class that assuming $\sum_{n=1}^{\infty} |f(n)| < \infty$ is one hypothesis that is sufficient to imply $(*)$.

(a) Prove that assuming

$$\prod_p \big(1 + |f(p)| + |f(p^2)| + \cdots\big) < \infty$$

is also sufficient to imply $(*)$.

(b) Show that assuming

$$\prod_p \big(1 + |f(p) + f(p^2) + \cdots|\big) < \infty$$

is *not* sufficient to imply $(*)$.

IV. In class we showed that

$$\sum_{1<n\le x} \frac{\phi(n)}{n^2 \log n} = \frac{6}{\pi^2} \log\log x + O(1).$$

By being more careful, find an explicit constant C and an explicit function $\varepsilon(x)$, tending to zero as $x \to \infty$, such that

$$\sum_{1<n\le x} \frac{\phi(n)}{n^2 \log n} = \frac{6}{\pi^2} \log\log x + C + O(\varepsilon(x)).$$

(Hint: define

$$E(x) = \sum_{n\le x} \frac{\phi(n)}{n} - \frac{6x}{\pi^2},$$

and don't estimate $E(x)$ as $O(\log x)$ until the end of the application of partial summation. Your constant should be, at least in part, a convergent integral with $E(x)$ in the integrand.)

V. (a) Let $f(t)$ be a function with continuous second derivative on $(0, \infty)$, and define $B_1(t) = t - \frac{1}{2}$ and $B_2(t) = t^2 - t + \frac{1}{6}$. Show that

$$\sum_{n\le x} f(n) = \int_1^x f(t)\,dt - B_1(\{x\})f(x) + \tfrac{1}{2}f(1) + \tfrac{1}{2}B_2(\{x\})f'(x) - \tfrac{1}{12}f'(1) - \tfrac{1}{2}\int_1^x B_2(\{t\})f''(t)\,dt.$$

(Hint: derive $\sum_{n\le x} f(n) = \int_{1-}^{x} f(t)\,d\big(t - B_1(\{t\})\big)$ and start summing by parts.)

(b) Show that

$$\sum_{n\le x} \frac{1}{n} = \log x + \gamma - \frac{B_1(\{x\})}{x} + O\left(\frac{1}{x^2}\right).$$

(c) Prove the following form of "Stirling's formula": for some constant c,

$$n! = \frac{n^n}{e^n}\sqrt{cn}\left(1 + O\left(\frac{1}{n}\right)\right). \qquad (**)$$

(Hint: consider $\sum_{m\le n} \log m$.)

(d) Assuming the truth of "Wallis's formula"

$$\frac{2}{\pi} = \prod_{n=1}^{\infty}\left(1 - \frac{1}{4n^2}\right) = \frac{1\times 3}{2\times 2} \times \frac{3\times 5}{4\times 4} \times \frac{5\times 7}{6\times 6} \times \cdots,$$

show that $c = 2\pi$ in the asymptotic formula $(**)$.

DEFINITION. The *generalized divisor function* $\tau_k(n)$ is defined, for any positive integer k, to be the number of ordered k-tuples $(d_1, \dots, d_k)$ of positive integers such that $d_1 \times \cdots \times d_k = n$, so that $\tau_2 = \tau$, for example.

VI. (a) Prove that $\tau_j * \tau_k = \tau_{j+k}$ for all positive integers j and k.

(b) Given the above relationship, what do you think a sensible way to define $\tau_{1/2}$ would be? What is $\tau_{1/2}(539)$? $\tau_{1/2}(16)$?

VII. Consider the following statement: "An integer that is relatively prime to q has, on average, $\sum_{p|q} 1/p$ fewer distinct prime factors than a random integer of the same size". Formulate this statement precisely and prove it. (If you want to, state and prove the analogous statement when the word "distinct" is removed.)

VIII. (a) Show that
$$\int_0^\infty \frac{\sin x}{x^\alpha}$$
converges for all $0 < \alpha \le 2$. (You don't have to determine the value of the integral. Hint: use calculus, as opposed to residues or anything complex.)

(b) Given a positive integer q, suppose that χ is an arithmetic function such that
$$\sum_{n=k}^{k+q-1} \chi(n) = 0$$
for every natural number k. Prove that
$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n^\beta}$$
converges for every $\beta > 0$. (Again, you don't have to determine the value of the sum.)

IX. Prove that $\displaystyle\sum_{n\le x} \frac{n}{\phi(n)} = \frac{\zeta(2)\zeta(3)}{\zeta(6)} x + O_\varepsilon(x^\varepsilon)$ for every $\varepsilon > 0$.

DEFINITION. A positive integer n is *powerfull* if any prime that divides n divides it to at least the second power: $p \mid n \Rightarrow p^2 \mid n$. (Hence "power-full", as opposed to "powerful".) For example, perfect powers are powerfull numbers, and the first powerfull number that is not a perfect power is $72 = 2^3 3^2$.

X. Define the following arithmetic functions:
$$f(n) = \begin{cases} 1, & \text{if } n \text{ is powerfull,} \\ 0, & \text{otherwise,} \end{cases} \qquad s(n) = \begin{cases} \mu(m), & \text{if } n = m^6, \\ 0, & \text{otherwise,} \end{cases}$$
$$r(n) = \#\{(a,b) \in \mathbb{N} \times \mathbb{N} \colon a^2 b^3 = n\}.$$

(a) Prove that $f = r * s$.

(b) Prove that
$$\sum_{n\le x} r(n) = \zeta\big(\tfrac{3}{2}\big) x^{1/2} + \zeta\big(\tfrac{2}{3}\big) x^{1/3} + O\big(x^{1/5}\big).$$

(c) Prove that the number of powerfull integers up to x is
$$\frac{\zeta\big(\frac{3}{2}\big)}{\zeta(3)} x^{1/2} + \frac{\zeta\big(\frac{2}{3}\big)}{\zeta(2)} x^{1/3} + O\big(x^{1/5}\big).$$