Math 539
Homework #3
due Wednesday, November 2, 2005 at 10 AM

Reality check problems. Not to write up; just ensure that you know how to do them.

I. Prove that

$$\prod_{p \le x} \left(1 - \frac{1}{p}\right) = \frac{1}{e^\gamma \log x} + O\left(\frac{1}{\log^2 x}\right).$$

II. Prove that

$$\theta(x) = \psi(x) + O(\sqrt{x}) \quad \text{and} \quad \pi(x) = \frac{\psi(x)}{\log x}\left(1 + O\left(\frac{1}{\log x}\right)\right).$$

Homework problems. To write up and hand in.

I. (a) Prove that

$$\sigma(n) \le \frac{\pi^2 e^\gamma}{6} n \log\log n + O(n),$$

showing that the main term is best possible.

(b) Prove that

$$\log \tau(n) \le \log 2 \cdot \frac{\log n}{\log\log n} + O\left(\frac{\log n}{\log\log^2 n}\right),$$

showing that the main term is best possible.

II. (a) Prove that $2^{\omega(n)} \le \tau(n) \le 2^{\Omega(n)}$ for all $n \ge 1$. What are the cases of equality for each inequality?

(b) We have seen that both $\omega(n) \sim \log\log n$ and $\Omega(n) \sim \log\log n$ hold for almost all integers n. Therefore by part (a), it seems we could conclude that $\tau(n) \sim 2^{\log\log n} = (\log n)^{\log 2}$ for almost all integers n. Why is this deduction invalid?

(c) Show that the correct deduction is that $\tau(n) = (\log n)^{\log 2 + o(1)}$ for almost all integers n. We have also seen that the average order of $\tau(n)$ is $\log n$. How can these two facts be reconciled?

III. Suppose that f is an arithmetic function whose average value exists and is nonzero. Prove that you can "change f at one prime" to make the average value anything you want. More specifically, prove that for any prime p and any $A \in \mathbb{R}$, there is an arithmetic function g whose average value is A that satisfies $g(n) = f(n)$ for every n with $(p, n) = 1$. (Note: it is not assumed that f is multiplicative.)

IV. In place of the trivial (but effective!) choice $D = \{1, 2\}$, $a_1 = 1$, $a_2 = -2$, Chebyshev chose $D = \{1, 2, 3, 5, 30\}$ and $a_1 = a_{30} = 1$, $a_2 = a_3 = a_5 = -1$. Using this choice or your own, compute upper and lower bounds for $\psi(x)$ using Chebyshev's method. For this problem, instead of using O-notation, keep *explicit constants* in all your upper and lower bounds.

V. (a) Verify that $\psi(x) \geq \theta(x) \geq \psi(x) - 2\psi(\sqrt{x})$.
   (b) Using the Chebyshev-type bounds on $\psi(x)$ from the previous problem, obtain explicit upper and lower bounds for $\theta(x)$. From these bounds, derive "Bertrand's postulate": for every $x \geq 2$, there is always a prime in the interval $[x, 2x]$.
   (c) What is the smallest constant K for which "there is always a prime in $[x, Kx]$" is true for every $x \geq 2$? Prove it.

VI. Rényi proved the following beautiful theorem about the set S_k of numbers n for which $\Omega(n) - \omega(n) = k$: the set S_k has a natural density d_k, and d_k is the coefficient of z_k in the power series expansion around 0 of the function

$$F(z) = \prod_p \left(1 - \frac{1}{p}\right)\left(1 + \frac{1}{p - z}\right).$$

Confirm that Rényi's theorem gives $6/\pi^2$ as the density of squarefree numbers. What is the density of the numbers that are squarefree except for a single squared prime factor?

VII. By "the $m \times n$ multiplication table" we mean the $m \times n$ array whose (i, j)-th entry is ij. Note that the $m \times n$ multiplication table has mn entries, each of which is a positive integer not exceeding mn, but there are repetitions due to commutativity and to multiple factorizations of various entries. Define $D(m, n)$ to be the number of *distinct* integers in the $m \times n$ multiplication table.
   (a) Erdős gave an ingenious argument showing that $D(n, n) = o(n^2)$. The idea is as follows: by the Erdős-Kac Theorem, almost all integers up to n have about $\log \log n$ prime factors (counted with mutiplicity), That means that almost all of the entries in the $n \times n$ multiplication table have about $2 \log \log n$ prime factors. But these entries do not exceed n^2, and almost all integers up to n^2 only have about $\log \log n^2 = \log \log n + \log 2$ prime factors. Therefore almost all integers up to n^2 must be missing from the $n \times n$ multiplication table. Turn this sketch into a rigorous proof. What is the smallest function $f(n)$ for which you can prove that $D(n, n) \ll f(n)$?
   (b) Generalize as best as you can to $D(m, n)$. How small can m be as a function of n so that $D(m, n) = o(mn)$ still?

VIII. Find the smallest positive integer n such that $\phi(6n + 1) < \phi(6n)$. It is probably a valuable hint to know that I have a paper titled *The smallest solution of $\phi(30n + 1) < \phi(30n)$ is ...* (`http://www.math.ubc.ca/~gerg/papers/abstracts/SS.html`). You are certainly allowed to use a computer for this problem; simply report enough detail to convince me that you did indeed fully derive the answer.

IX. Define $\Delta(x) = \sum_{n \le x} \tau(n) - (x \log x + (2\gamma - 1)x)$.

(a) Show that

$$\Delta(x) = -2 \sum_{n \le \sqrt{x}} B_1\left(\left\{\frac{x}{n}\right\}\right) + O(1),$$

where B_1 was defined on Homework #2.

(b) Prove that

$$\int_0^x \Delta(t)\, dt \ll x.$$

(c) Conclude that

$$\sum_{n \le x} \tau(n)\left(1 - \frac{n}{x}\right) = \frac{1}{x}\int_0^x \left(\sum_{n \le t} \tau(n)\right) dt = \tfrac{1}{2} x \log x + \left(\gamma - \tfrac{3}{4}\right) + O(1).$$

Does this imply that the "right" average order for $\tau(n)$ is $\log x + 2\gamma - \frac{3}{2}$ instead of $\log x + 2\gamma - 1$?

Open-ended problem. To write up and hand in if you want to ("extra credit").

X. In my paper *Dimensions of the spaces of cusp forms and newforms on* $\Gamma_0(N)$ *and* $\Gamma_1(N)$ (`http://www.math.ubc.ca/~gerg/papers/abstracts/DSCFN.html`), I found that certain functions related by convolutions had average values that differed by factors of the form $\zeta(2)$ and $\zeta(3)$. See Theorems 8 and 9 for the average orders, and see Theorems 1, 4, 13, and 14 and Propositions 11 and 15 for formulas for the g functions; in each case, everything except the s function winds up in the error term, so Theorems 8 and 9 are really statements about the average order of the s functions.

This motivates the following problem: Given an arithmetic function f (you may assume multiplicativity if you like) whose average value is C, formulate conditions under which the convolution $f * \tau_k$ has average value $C\zeta(m)^k$ for all $k \in \mathbb{N}$. (Here m can depend upon the conditions you formulate.) Does your statement hold for $k \in -\mathbb{N}$ as well? What is the interpretation of τ_k for negative integers k?