Math 539
Homework #4
due Wednesday, November 16, 2005 at 10 AM

Reality check problems. Not to write up; just ensure that you know how to do them.

I. Let $A(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series with $a_n \geq 0$ for all $n \in \mathbb{N}$. Show that $\sigma_a = \sigma_c$ and that
$$\sup_{t\in\mathbb{R}} |A(\sigma + it)| = A(\sigma)$$
for all $\sigma > \sigma_c$.

II. Finish the verification that
$$0 < \frac{1}{2} - \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{ds}{s} < \frac{c}{\pi T}$$
for $c, T > 0$.

III. Acquire a working knowledge of Appendices A.2–A.4 of Bateman and Diamond (pp. 346–352).

IV. Acquire a working knowledge of the attached appendix on the Bernoulli polynomials $B_k(x)$, the Bernoulli numbers B_k, and the Euler-Maclaurin summation. (As the nomenclature suggests, this material is classical; the attached notes in particular are from an earlier draft of the soon-to-be-published *A Primer in Multiplicative Number Theory, Volume I* written by Montgomery and Vaughan.)

Homework problems. To write up and hand in.

I. Bateman and Diamond, p. 122, #6.18. (Hint: partial summation.)

II. Bateman and Diamond, pp. 128–129, #6.21

III. Let $x > 1$ be fixed. Calculate the residue of the meromorphic function $\zeta^2(s)x^s/s$ at $s = 1$. (Hint: the answer is *not* $2\gamma x$.) Express appropriate excitement over the result.

IV. Given a Dirichlet series $A(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ that is not identically zero, define F to be the smallest number such that $a_F \neq 0$. Show rigorously that $A(s) \sim a_F F^{-s}$ as $\sigma \to \infty$.

V. Suppose that $A(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ is a Dirichlet series such that $A(0)$ converges, and define $R(x) = \sum_{n>x} a_n$. Show that the abscissa of convergence for $A(s)$ is the infemum of those real numbers θ for which $R(x) \ll_\theta x^\theta$.

VI. Show that both $\sum_{n=1}^{\infty} \sigma(n^2)n^{-s}$ and $\sum_{n=1}^{\infty} \sigma(n)^2 n^{-s}$ can be written in terms of the Riemann zeta function.

VII. Show that $\sigma_a - \sigma_c$, the difference between the abscissa of absolute convergence of a Dirichlet series and its abscissa of convergence, can take any value in $[0, 1]$. (Hint: consider linear combinations of shifts of $\zeta(s)$ and $\eta(s) = \sum_{n=1}^{\infty}(-1)^{n-1}n^{-s}$.)

DEFINITION. An arithmetic function f is *of polynomial growth* if there are positive constants C, d such that $|f(n)| \le Cn^d$ for all $n \in \mathbb{N}$.

VIII. Recall Theorem 2.7: any arithmetic function h with $h(1) \neq 0$ has a convolution inverse h^{-1} satisfying $h * h^{-1} = e$.

(a) Suppose that g is an arithmetic function satisfying $g(1) = 1$ and $|g(n)| \le 1$ for all $n \in \mathbb{N}$. Prove that $|g^{-1}(n)| \le n^2$ for all $n \in \mathbb{N}$. (Hint: use $g * g^{-1} = e$ to bound $g^{-1}(n)$ recursively.)

(b) If f is an arithmetic function of polynomial growth with $f(1) \neq 0$, prove that f^{-1} is also of polynomial growth.

(c) Suppose that $A(s)$ is a Dirichlet series that converges in some right half-plane. Show that $1/A(s)$ is represented by a Dirichlet series that converges in some right half-plane if and only if $\lim_{\sigma\to\infty} A(s) \neq 0$.

IX. Given $k \in \mathbb{N}$, define $\Omega_k(n)$ to be the total number of prime factors of n (with multiplicity), but only counting primes that are larger than k:

$$\Omega_k(n) = \sum_{\substack{p^r \| n \\ p>k}} r.$$

Let $E_k(s) = \sum_{n=1}^{\infty} k^{\Omega_k(n)} n^{-s}$. Prove that $E_k(s) = \zeta^k(s) F_k(s)$, where F_k is a Dirichlet series that converges for $\sigma > \frac{1}{2}$. Conclude that

$$\lim_{s\to 1} (s-1)^k E_k(s) = F_k(1) = \prod_{p>k} \left(1 - \frac{k}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^{k}.$$

What goes wrong if we replace $\Omega_k(n)$ with $\Omega(n)$? Why should we expect this? How does the limit change if we replace $\Omega_k(n)$ by $\omega(n)$ in the definition of E_k?

X. (a) Let N and K be any positive integers. Show that for $\sigma > 1$,

$$\zeta(s) = \sum_{n\le N} n^{-s} + \frac{N^{1-s}}{s-1} + N^{-s} \sum_{k=1}^{K} \binom{s+k-2}{k-1} \frac{B_k}{kN^{k-1}} - \binom{s+K-1}{K} \int_N^{\infty} B_K(\{x\}) x^{-s-K}\, dx,$$

where B_k is the kth Bernoulli number, B_K is the Kth Bernoulli polynomial, and for any complex number z and positive integer m the binomial coefficient $\binom{z}{m}$ is defined to be the polynomial value $\binom{z}{m} = z(z-1)\dots(z-m+1)/m!$. (Hint: use Euler-Maclaurin summation on the sum $\sum_{n>N} n^{-s}$.)

(b) Show that $\zeta(s)$ can be analytically continued to a meromorphic function on the entire complex plane, with its only pole a simple pole at $s = 1$. (Hint: you don't have to get the entire complex plane in one go—bigger and bigger pieces of it will suffice.)

(c) Compute the value of $\zeta(n)$ on every *nonpositive* integer n. (Hint: take $N = 1$ and $s = 2 - K$ in the equation from part (a). Setting $x = -1$ in equations (5) and (15) of the attached appendix will also provide useful information.)

Bonus problem: Using problem X(a) or otherwise, calculate $\zeta'(0)$ in closed form.