

Math 539 Homework #1

due Friday, January 15, 2010 at the beginning of class

Homework policies: You are permitted to consult one another concerning the homework assignments, but your submitted solutions must be written by you in your own words. I will consider not only correctness but also clarity when evaluating your work.

Open invitation: From time to time in the lectures, I leave particular results as “Exercises” for you; doing those exercises is an excellent way to ensure that you are grasping the material. You don’t need to write those Exercises up. However, if you ever have questions about them, I invite you to write the questions somewhere in your homework solutions; I will try to answer them in writing or in class. (Of course, if you write a question that’s about the course but not specifically about the Exercises, I’ll probably answer those too....)

- I. Let $f(n)$ be a multiplicative function. For any fixed positive integer q such that $f(q) \neq 0$, prove that the function $g(n) = f(qn)/f(q)$ is a multiplicative function of n .
- II. Let $\{q_1, q_2, \dots\} = \{2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 17, \dots\}$ denote the sequence of all prime powers. Let $Q(n)$ denote the largest prime power dividing n .
- (a) Prove that $\lim_{n \rightarrow \infty} Q(n) = \infty$.
 - (b) Suppose that $f(n)$ is a multiplicative function with the property that $\lim_{k \rightarrow \infty} f(q_k) = 0$. Must it be true that $\lim_{n \rightarrow \infty} f(n) = 0$ as well?
- III. Let h and k be functions with $h(x) > 2$ and $k(x) > 2$ for all sufficiently large x .
- (a) Prove that $h(x) \ll k(x)$ implies $\log h(x) \ll \log k(x)$.
 - (b) Show via a counterexample that the part (a) is false if we only require $k(x) > 1$ for sufficiently large x .
 - (c) Show via a counterexample that the converse to part (a) is false.
 - (d) Prove that $\log h(x) = o(\log k(x))$ implies $h(x) = o(k(x))$.
 - (e) For all real numbers $A > 0$ and $0 < b < 1$ and $\varepsilon > 0$, show that

$$\log^A x \ll \exp(\log^b x) \ll x^\varepsilon$$

uniformly for $x \geq 1$.

- IV. Let $\log_k n$ denote the k -fold iterated logarithm $\log \log \dots \log n$. Prove that for every $k \geq 1$ and every $\varepsilon > 0$,

$$\sum_{n: \log_k n > 0} \frac{1}{n(\log n)(\log \log n) \cdots (\log_{k-1} n)(\log_k n)^{1+\varepsilon}} \text{ converges}$$

while

$$\sum_{n: \log_k n > 0} \frac{1}{n(\log n)(\log \log n) \cdots (\log_{k-1} n)(\log_k n)^{1-\varepsilon}} \text{ diverges.}$$

(In particular, this shows that $\sum_{n \geq 2} 1/n(\log n)^{1+\varepsilon}$ converges while $\sum_{n \geq 2} 1/n(\log n)^{1-\varepsilon}$ diverges.)

(continued on next page)

V. Let f be a multiplicative function. We would like to have conditions under which we can conclude that both expressions

$$\sum_{n=1}^{\infty} f(n) \quad \text{and} \quad \prod_p (1 + f(p) + f(p^2) + \cdots) \quad \text{converge to equal values.} \quad (*)$$

We know (Theorem 1.9) that assuming $\sum_{n=1}^{\infty} |f(n)| < \infty$ is one hypothesis that is sufficient to imply (*).

(a) Prove that assuming

$$\prod_p (1 + |f(p)| + |f(p^2)| + \cdots) < \infty$$

is also sufficient to imply (*).

(b) Show that assuming

$$\prod_p (1 + |f(p) + f(p^2) + \cdots|) < \infty$$

is *not* sufficient to imply (*).

VI. Let $s(x)$ be any function defined on the interval $[0, 1]$, and define

$$F(n) = \sum_{1 \leq k \leq n} s\left(\frac{k}{n}\right) \quad \text{and} \quad G(n) = \sum_{\substack{1 \leq k \leq n \\ (k,n)=1}} s\left(\frac{k}{n}\right).$$

(a) Prove that $G = F * \mu$.

(b) Evaluate the sum of the primitive n th roots of unity

$$\sum_{\substack{1 \leq k \leq n \\ (k,n)=1}} e^{2\pi i k/n}$$

as a function of n .

VII. Prove that the following identities all hold in suitable half-planes (be explicit about which half-planes).

$$(a) \quad \sum_{\substack{n \geq 1 \\ (n,q)=1}} \frac{1}{n^s} = \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right) \quad \text{for any positive integer } q$$

$$(b) \quad \sum_{n=1}^{\infty} \frac{d(n^2)}{n^s} = \frac{\zeta(s)^3}{\zeta(2s)}$$

$$(c) \quad \sum_{n=1}^{\infty} \frac{\sigma_a(n)}{n^s} = \zeta(s)\zeta(s-a) \quad \text{for any complex number } a, \text{ where } \sigma_a(n) = \sum_{d|n} d^a \text{ is the generalized sum-of-divisors function.}$$

(continued on next page)

- VIII. Define the Dirichlet series $P(s) = \sum_p \frac{1}{p^s}$ and $W(s) = \sum_{n=1}^{\infty} \frac{\omega(n)}{n^s}$, where $\omega(n)$ is the number of distinct prime factors of n .
- What is the abscissa of convergence for $P(s)$?
 - Prove that formally, $W(s) = \zeta(s)P(s)$.
 - What is the abscissa of convergence for $W(s)$?
 - Can $W(s)$ be analytically continued to an entire function?
 - Prove that $\sum_{n \leq x} \omega(n) \ll_{\varepsilon} x^{1+\varepsilon}$ for every $\varepsilon > 0$. (You may assume Theorem 1.3.)
- IX. Montgomery and Vaughan, Section 1.1, p. 9, #6(a)–(c). Hint for (b): consider $\sum_F z^{\deg F}$, where the sum is taken over all monic polynomials in $\mathbb{F}_p[x]$.
- X. Montgomery and Vaughan, Section 1.2, p. 18, #5
- XI. (a) Montgomery and Vaughan, Section 1.3, pp. 28–29, #11(b)–(c). Note the required region $\sigma > 0$ in (b).
- Show that $\sum_{n=1}^{\infty} (-1)^n n^{-1+2010\pi i/\log 2} = 0$. (In analytic number theory, \log always denotes the natural logarithm.)
- XII. The *generalized divisor function* $d_k(n)$ is defined, for any positive integer k , to be the number of ordered k -tuples (m_1, \dots, m_k) of positive integers such that $m_1 \times \dots \times m_k = n$, so that $d_2(n) = d(n)$, for example.
- Prove that $d_j * d_k = d_{j+k}$ for all positive integers j and k . Given this relationship, what do you think a sensible way to define $d_{1/2}$ would be? Calculate $d_{1/2}(539)$ and $d_{1/2}(16)$. Can you write down a formula for $d_{1/2}(p^r)$ as a function of r ?
 - Prove that $\sum_{n=1}^{\infty} d_k(n)n^{-s} = \zeta(s)^k$, in a suitable half-plane, for all positive integers k . Given this relationship, what do you think a sensible way to define d_z would be for any complex number z ? Calculate $d_i(539)$ and $d_i(16)$ (where $i = \sqrt{-1}$).