Math 539 Homework #2

due Friday, January 29, 2010 at the beginning of class

I. Recall that $\mathrm{li}(x) = \int_2^x \frac{dt}{\log t}$ and that $\pi(x) = \#\{p \le x \colon p \text{ is prime}\}$.

(a) Let K be a positive integer. Show that

$$\mathrm{li}(x) = \frac{x}{\log x} + \frac{x}{\log^2 x} + 2\frac{x}{\log^3 x} + \cdots + (K-1)!\frac{x}{\log^K x} + O_K\left(\frac{x}{(\log x)^{K+1}}\right).$$

Hint: you can estimate an integral by splitting the interval of integration at some middle point.

(b) Starting from Mertens's formula (Montgomery and Vaughan, Theorem 2.7), use partial summation to see what can be deduced about $\pi(x)$ in this way. Can you prove in this way that $\pi(x) \ll \mathrm{li}(x)$? that $\pi(x) \gg \mathrm{li}(x)$? that $\pi(x) \sim \mathrm{li}(x)$? (The answer is "yes" for at least one of these, and "no" for at least one.)

II. Define $\Phi(s) = \sum_{n-1}^{\infty} \phi(n) n^{-s}$.

(a) Show directly from the definition of σ_c that the abscissa of convergence of $\Phi(s)$ is $\sigma_c = 2$.

(b) For $\sigma > 2$, write $\Phi(s)$ in terms of the Riemann zeta function.

III. Define $F(x) = \sum_{n \le x} \frac{\phi(n)}{n}$ and $Q(x) = \sum_{n \le x} \mu^2(n)$. (Note: $Q(x)$ is the number of squarefree integers not exceeding x.) Let y be a real number in the range $\log^2 x \le y \le x$.

(a) Show that $F(x) - F(x-y) \sim \frac{6}{\pi^2} y$. (Hint: use the asymptotic formula we already know for $F(x)$.) In other words, the average value of $\frac{\phi(n)}{n}$ is $\frac{6}{\pi^2}$ even over intervals around x as short as $\log^2 x$.

(b) Why does the same approach fail to prove that $Q(x) - Q(x-y) \sim \frac{6}{\pi^2} y$?

IV. Let f be an arithmetic function, and let σ_a be the abscissa of absolute convergence of its Dirichlet series $\sum_{n=1}^{\infty} f(n) n^{-s}$. Let $\varepsilon > 0$ be a constant.

(a) If $\sigma_a \ge 0$, show that $\sum_{d \le x} |f(d)| \ll_\varepsilon x^{\sigma_a + \varepsilon}$.

(b) If $\sigma_a < 1$, show that $\sum_{d > x} |f(d)|/d \ll_\varepsilon x^{\sigma_a - 1 + \varepsilon}$.

V. (a) Find the smallest constant S such that $\sigma(n) < Sn \log\log n + O(n)$ for all positive integers n.

(b) Find the smallest constant T such that $\sigma(n) \le T n^{21/20}$ for all positive integers n.

(c) Are there finitely many or infinitely many positive integers n for which $\sigma(n) \ge n^{21/20}$?

VI. Define the *logarithmic density* of a set S of integers to be the following limit (if it exists):

$$\lim_{x \to \infty} \frac{1}{\log x} \sum_{\substack{n \le x \\ n \in S}} \frac{1}{n}.$$

Let S_3 be the set of all positive integers whose first (leftmost) digit is 3.

(a) Suppose that the (regular) density of the set S exists and equals c. Show that the logarithmic density of S also exists and equals c.

(b) Show that the density of S_3 does not exist.

(c) Show that the logarithmic density of S_3 does exist, and calculate it.

(continued on next page)

VII. (a) Prove that

$$\sum_{m\le x}\sum_{\substack{n\le x\\(m,n)=1}} 1 = \sum_{d\le x}\mu(d)\left\lfloor\frac{x}{d}\right\rfloor^2.$$

Hint: what does $\sum_{d|(m,n)}\mu(d)$ equal?

(b) Write down the rigorous definition of what a number theorist refers to as "the probability that two randomly chosen integers are relatively prime to each other", and calculate it. (Remark: you should be able to see that this is the same question as "the probability that a randomly chosen lattice point does not have any other lattice points on the line segment between it and the origin".)

VIII. (a) Prove that the average value of $n/\phi(n)$ is $\zeta(2)\zeta(3)/\zeta(6)$.

(b) Let p be a prime, and let $\nu_p(n)$ denote the power of p in the factorization of n; for example, $\nu_3(8)=0$, $\nu_3(12)=1$, and $\nu_3(18)=2$. Prove that the average value of $\nu_p(n)$ is $1/(p-1)$.

IX. (a) What is wrong with the following beginning of an attempt to investigate the sum $\sum_{n\le x}\omega(n)^2$?

$$\sum_{n\le x}\omega(n)^2 = \sum_{n\le x}\Big(\sum_{p|n}1\Big)^2 = \sum_{n\le x}\sum_{p_1|n}\sum_{p_2|n}1 = \sum_{p_1\le x}\sum_{p_2\le x}\sum_{\substack{n\le x\\p_1p_2|n}}1 = \sum_{p_1\le x}\sum_{p_2\le x}\left\lfloor\frac{x}{p_1p_2}\right\rfloor.$$

(b) Correct this beginning of an attempt, and find an asymptotic formula (with error term) for $\sum_{n\le x}\omega^2(n)$.

X. Montgomery and Vaughan, Section 2.3, p. 63, #6

XI. Find an asymptotic formula (with error term) for $\sum_{n\le x} d_3(n)$.

XII. By "the $n\times n$ multiplication table" we mean the $n\times n$ array whose (i,j)-th entry is ij. Note that the $n\times n$ multiplication table has n^2 entries, each of which is a positive integer not exceeding n^2, but there are repetitions due to commutativity and to multiple factorizations of various entries.

Define $D(n)$ to be the number of *distinct* integers in the $n\times n$ multiplication table. Erdős gave an ingenious argument showing that $D(n)=o(n^2)$. The idea is as follows: by the Hardy–Ramanujan Theorem, almost all integers up to n have about $\log\log n$ prime factors. That means that almost all of the entries in the $n\times n$ multiplication table have about $2\log\log n$ prime factors. But these entries do not exceed n^2, and almost all integers up to n^2 only have about $\log\log n^2=\log\log n+\log 2$ prime factors. Therefore almost all integers up to n^2 must be missing from the $n\times n$ multiplication table.

Turn this sketch into a rigorous, quantitative proof: find an explicit function $f(n)$, satisfying $f(n)=o(n^2)$, for which you can prove that $D(n)\ll f(n)$.

Bonus: Montgomery and Vaughan, Section 2.1, p. 41, #10. This problem is fully optional for you. (I tried for a little bit to do this one, but I couldn't. Can someone enlighten me?)