Math 539 Homework #3

due Friday, February 12, 2010 at the beginning of class

I. Suppose that $1 \le a \le q$ and $(a,q)=1$. Prove that
$$\sum_{\substack{p\le x\\ p\equiv a \pmod q}} \frac1p \ll \frac1a + \frac{\log\log x}{\phi(q)}$$
and that the term $\frac1a$ can be omitted if a is not a prime. The $\ll$-constant should be absolute (that is, independent of a and q). Hint: use partial summation and the Brun-Titchmarsh inequality.

II. Let $q\in\mathbb{N}$, let f and g be two real-valued functions defined on $\mathbb{Z}/q\mathbb{Z}$, and let g_r be defined by $g_r(n)=g(-n)$. Let $f\star g$ (not to be confused with the Dirichlet convolution $f*g$ of two arithmetic functions) denote the function defined on $\mathbb{Z}/q\mathbb{Z}$ by
$$(f\star g)(n) = \frac1q \sum_{a \pmod q} f(a)g(n-a).$$
(a) Prove that $\widehat{f_r}(k) = \overline{\hat f(k)}$.
(b) Prove that $\widehat{(f\star g)}(k) = \hat f(k)\hat g(k)$.
(c) Define $h = f\star(f+f_r)$. Prove that $\hat h(k)$ has nonnegative real part for every $k\in\mathbb{Z}/q\mathbb{Z}$.

III. Let p be a prime, and let δ denote the characteristic function of the set of integers that are primitive roots (mod p).
(a) Explain why we immediately know that there exist constants c_χ such that $\delta(n) = \sum_{\chi \pmod p} c_\chi \chi(n)$ for every integer n.
(b) In the notation of part (a), prove that $c_\chi = \frac{1}{p-1}\sum_g \chi(g)$, where the sum is taken over all primitive roots g (mod p). Note: I do mean $\chi(g)$, not $\overline{\chi(g)}$.
(c) BONUS: Prove that if χ has order b in the character group (mod p), then $c_\chi = \dfrac{\phi(p-1)}{p-1}\dfrac{\mu(b)}{\phi(b)}$.

IV. (a) Write down all Dirichlet characters to the modulus 15; to the modulus 16. Indicate which of them are primitive characters.
(b) Montgomery and Vaughan, Section 9.1, p. 285, #5

V. (this problem has been intentionally left blank)

VI. Montgomery and Vaughan, Section 4.2, p. 119, #2 and 3. Hint: a sum of the form $\sum_y \big|\sum_z \cdots\big|^2$ can be written as a triple sum $\sum_y \big(\sum_{z_1}\cdots\big)\overline{\big(\sum_{z_2}\cdots\big)}$.

VII. Montgomery and Vaughan, Section 4.2, pp. 119–120, #4 and 5

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VIII. Let χ denote a non-principal Dirichlet character.
   (a) Define $S_\chi(x) = \sum_{n \le x} \chi(n)$. Prove that $S_\chi(x) \ll_\chi 1$. Hint: use the periodicity of χ.
   (b) Montgomery and Vaughan, Section 4.3, p. 127, #2

IX. Montgomery and Vaughan, Section 4.3, pp. 128–129, #5(f). You may use the results from Section 4.3 (state what you're using); if you want to use anything from #5(a)–(e), you should prove it.

X. (this problem has been intentionally left blank)

XI. Montgomery and Vaughan, Section 9.1, pp. 285–286, #6

XII. For every finite abelian group G, show that there exist infinitely integers n for which G is isomorphic to a subgroup of $(\mathbb{Z}/n\mathbb{Z})^\times$.