Math 539 Homework #4

due Monday, March 15, 2010 at the beginning of class

I. Show that if $\sigma_c < \sigma_0 < 0$, then
$$\lim_{T\to\infty} \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \alpha(s) \frac{x^s}{s}\, ds = -{\sum_{n>x}}' a_n.$$
(Hint: start by adapting equation (5.3) to the case where $\sigma_0 < 0$, then use the method of proof of Theorem 5.1.) This problem was originally Montgomery and Vaughan, Section 5.1, p. 145, #1, but there was a negative sign missing.

II. Show that it is necessary to invoke a smaller radius r in Jensen's inequality, in the following sense: Suppose that f is analytic on the ball $B_R = \{z \in \mathbb{C} \colon |z| \le R\}$, with $f(0) \ne 0$ and $|f(z)| \le M$ in B_R. Show by example that *it is not necessarily true* that the number of zeros of f in B_R is $\ll_R \log\big(M/|f(0)|\big)$. (Hint: consider Blaschke products).

III. Montgomery and Vaughan, Section 6.1, p. 176, #4

IV. Assuming that $\theta(x) = x + O\big(x \exp(-c\sqrt{\log x})\big)$ for some positive constant c, prove by partial summation that $\pi(x) = \mathrm{li}(x) + O\big(x \exp(-c\sqrt{\log x})\big)$ for the same c.

V. Montgomery and Vaughan, Section 6.2, p. 182, #1 and 2

VI. Let p_n denote the nth prime number. Using the Prime Number Theorem, prove that
$$p_n = n\log n + n\log\log n - n + O\bigg(\frac{n\log\log n}{\log n}\bigg).$$

VII. Montgomery and Vaughan, Section 6.2, p. 184, #11(b)

VIII. Montgomery and Vaughan, Section 6.2, p. 185, #16(c)

IX. (a) By dividing the Laurent expensions of $\zeta'(s+1)$ and $\zeta(s+1)$, or otherwise, show that
$$\frac{\zeta'(s+1)}{\zeta(s+1)} = -\frac{1}{s} + C_0 + O(s)$$
near $s = 0$, where C_0 is Euler's constant.

(b) Montgomery and Vaughan, Section 6.2, p. 182, #4

X. Prove that
$$\sum_{n\le x} \phi(n) = \tfrac{3}{\pi^2}x^2 + O\big(x^2 \exp(-c\sqrt{\log x})\big),$$
where c is a small positive constant.

(continued on next page)

XI. Let $k \geq 2$ be an integer, and let $d_k(n)$ be the generalized divisor function (so that $d_2 = d$). *Give a heuristic argument* that there exists a polynomial $P_k(x)$ with real coefficients, of degree $k-1$ with leading coefficient $1/(k-1)!$, such that

$$\sum_{n \leq x} d_k(n) = x P_k(\log x) + o(x).$$

("Give a heuristic argument" means you can ignore all error terms along the way.)

XII. Define $F(s) = \sum_{n=1}^{\infty} \phi(n)^{-s}$.

(a) Show that $F(s)$ is analytic on the right half-plane $\sigma > 1$.

(b) Show that $F(s) = \zeta(s)Q(s)$ for $\sigma > 1$, where where

$$Q(s) = \prod_p \big(1 + (p-1)^{-s} - p^{-s}\big).$$

(c) Show that for any constant $0 < \varepsilon < 1$, the product defining $Q(s)$ converges to an analytic function on the right half-plane $\sigma \geq \varepsilon$ that is uniformly bounded on that half-plane.

(d) Let $f(m)$ be the number of positive integers n such that $\phi(n) = m$. Show that $F(s) = \sum_{m=1}^{\infty} f(m) m^{-s}$.

(e) *Give a heuristic argument* that

$$\#\{n \in \mathbb{N} \colon \phi(n) \leq x\} \sim \frac{\zeta(2)\zeta(3)}{\zeta(6)} x.$$

Why would part (c) of this problem be relevant if you were working through all the details to make your heuristic argument rigorous?