Math 539 Homework #5

due Monday, March 29, 2010 at the beginning of class

I. Consider the following arithmetic functions:

$$n\ ; \quad n^n\ ; \quad n!\ ; \quad 539^{\pi(n)}\ ; \quad \operatorname{lcm}\{1,2,\dots,n\}\ ; \quad \prod_{p\le n} p\ .$$

Label these functions as $f_1(n),\dots,f_6(n)$ in such a way that $f_1(n) \ll f_2(n) \ll \cdots \ll f_6(n)$. Then, put the consecutive ratios $f_2(n)/f_1(n),\dots,f_6(n)/f_5(n)$ in order of increasing magnitude as well.

II. Let $H_k = \sum_{m=1}^k \frac{1}{m}$ denote the kth harmonic number, and write $H_k = N_k/D_k$ in lowest terms (that is, N_k and D_k are relatively prime positive integers). We can determine how large N_k and D_k are by the following method.

(a) Using the prime number theorem, show that $\log D_k \le k + O(k/\log k)$ and $\log N_k \le k + O(k/\log k)$. Conclude that if a prime p divides N_k, then $\log p \le k + O(k/\log k)$.

(b) Suppose that $p \le k$ but $p \nmid D_k$. Prove that $p \le k/\log k + O(k/\log^2 k)$. (Hint: some cancellation has to occur if $p \nmid D_k$. Use part (a) to decide when that cancellation is possible.)

(c) Prove that when k is sufficiently large, every prime between $2k/\log k$ and k divides D_k. Conclude that $\log D_k = k + O(k/\log k)$ and $\log N_k = k + O(k/\log k)$.

III. (this problem has been intentionally left blank)

IV. Montgomery and Vaughan, Section 10.1, p. 335, #3(a). You may use, without proof, the formulas in the “Abelian weights” part of Section 5.1 of Montgomery and Vaughan.

V. Montgomery and Vaughan, Section 10.1, p. 336, #8, #9, and #11

VI. Montgomery and Vaughan, Section 10.1, p. 339, #19

VII. Montgomery and Vaughan, Section 10.2, p. 353, #2

VIII. (this problem has been intentionally left blank)

IX. Montgomery and Vaughan, Section 10.2, pp. 353–354, #4. Hint for part (a): use Thm. C.1.

X. Montgomery and Vaughan, Section 12.1, p. 409, #6. You may use, without proof, the formulas in the “Cesàro weights” part of Section 5.1 of Montgomery and Vaughan, although you might need to establish some variants where the integrals run from $c - iT$ to $c + iT$.

XI. (a) Given an integer m, show that

$$\sin \pi s = \pi(-1)^m (s-m)\big(1 + O\big(|s-m|^2\big)\big)$$

near $s = m$.

(b) Montgomery and Vaughan, Section 12.1, p. 409, #8(a)–(b). Hint for one solution to #8(a): use equation (C.6).

XII. Montgomery and Vaughan, Section 13.1, p. 430, #1