Math 539 Homework #6

due Wednesday, April 21, 2010 at the beginning of class

I. Montgomery and Vaughan, Section 10.1, p. 336, #13

II. Montgomery and Vaughan, Section 10.1, p. 342, #24. There is a negative sign missing on the right-hand side of part (c): it should say $L'(0,\chi) = -L(0,\chi)\log q + \cdots$. The Hurwitz zeta function $\zeta(s,\alpha)$ is defined in problem #22 on page 340. When solving this problem, you may use the conclusions in Montgomery and Vaughan, Section 10.1, pp. 340–342, #22–23 (just state clearly what you're using). Caution: do not overlook issues of convergence in part (a) of this problem.

III. Montgomery and Vaughan, Section 11.1, p. 366, #4. You should assume that χ is a *non-principal* character. Part (c) is a bonus: you can do part (c) for extra marks, or you can simply assume part (c) when doing parts (d) and (e).

IV. Montgomery and Vaughan, Section 11.2, p. 374, #1

V. Define $F(x) = \sum_{p\le x} \phi(p-1)/(p-1)$.

(a) Justify the identity

$$F(x) = \sum_{d\le x-1} \frac{\mu(d)}{d}\pi(x;d,1).$$

Hint: adapt the proof of Theorem 2.1.

(b) Prove that $F(x) = \alpha\,\mathrm{li}(x) + O_A\big(x/\log^A x\big)$ for every constant $A>0$, where

$$\alpha = \prod_p \left(1 - \frac{1}{p(p-1)}\right).$$

Hint: split the sum in part (a) at a suitable point, and use Corollary 11.21 for the small values of d and the trivial bound $\pi(x;d,1) \le x/d$ for the large values of d.

VI. Let $(a,q)=1$. In this problem, (lowercase) c is some convenient small positive constant.

(a) Given $\delta>0$, prove that there exists an *effective* constant $C(\delta)$ such that

$$\left|\psi(x;q,a) - \frac{x}{\phi(q)}\right| \le C(\delta)x\exp(-c\sqrt{\log x})$$

uniformly for $q \le (\log x)^{1-\delta}$. Hint: use Corollary 11.17 and Corollary 11.12.

(b) Prove that

$$\sum_{\substack{p\le x\\ p\equiv a \ (\mathrm{mod}\ q)}} \frac{1}{p} = \frac{\log\log x}{\phi(q)} + O_\delta(1)$$

uniformly for $q \le (\log x)^{1-\delta}$. Hint: explain why the left-hand side equals

$$\int_{2-}^{x} \frac{1}{t\log t}\, d\left(\frac{t}{\phi(q)} + \left(\theta(t;q,a) - \frac{t}{\phi(q)}\right)\right),$$

then use part (a). You might also find problem VIII below helpful.

(continued on next page)

VII. Define $\nu_2(m)$ to be the exponent of 2 in the prime factorization of m, so that $2^{\nu_2(m)} \parallel m$. Also define $M(x) = \sum_{n\le x} \nu_2(\phi(n))$.

(a) Show that

$$M(x) = \sum_{n\le x} \bigg(\max\{0, \nu_2(n) - 1\} + \sum_{p\mid n} \nu_2(p-1) \bigg).$$

(b) Show that

$$M(x) = x \sum_{k=1}^{\infty} k \bigg(\sum_{\substack{p\le x \\ p\equiv 2^k+1 \ (\mathrm{mod}\ 2^{k+1})}} \frac{1}{p} \bigg) + O(x).$$

(c) Prove that $M(x) = 2x\log\log x + O\big(x(\log\log\log x)^2\big)$, where the implicit O-constant is effective. Hint: split the sum over k in part (b) at a suitable point, and use problem VI above for the small values of k and problem I on Homework #3 for the large values of k.

VIII. Assume the Generalized Riemann Hypothesis for this problem.

(a) Let $(r, q) = 1$. Prove that

$$\psi(x; q, r) = \theta(x; q, r) + \frac{c_r(q)x^{1/2}}{\phi(q)} + O_q(x^{1/3}),$$

where $c_r(q)$ is the number of solutions to $x^2 \equiv r \ (\mathrm{mod}\ q)$.

(b) Let $(ab, q) = 1$, and suppose that the congruence $m^2 \equiv a \ (\mathrm{mod}\ q)$ has no solution but that the congruence $m^2 \equiv b \ (\mathrm{mod}\ q)$ has a solution. Prove that

$$\theta(x; q, a) - \theta(x; q, b) = \frac{c(q)x^{1/2}}{\phi(q)} - \frac{1}{\phi(q)} \sum_{\chi\ (\mathrm{mod}\ q)} \big(\bar\chi(a) - \bar\chi(b)\big) \sum_{\rho} \frac{x^\rho}{\rho} + O_q(x^{1/3}),$$

where $c(q)$ is the number of solutions to $x^2 \equiv 1 \ (\mathrm{mod}\ q)$; here the inner sum is over all nontrivial zeros ρ of $L(s,\chi)$. Hint: use Corollary 12.11.

IX. Let $S_\chi(x) = \sum_{n\le x} \chi(n)$, where $\chi \ (\mathrm{mod}\ q)$ is primitive and $q > 1$. Justify the identity

$$\tau(\chi) = \int_0^q e\big(\tfrac{t}{q}\big)\, dS_\chi(t),$$

and use it to prove that there exists a real number x such that $|S_\chi(x)| \ge \frac{1}{2\pi}\sqrt{q}$.

X. Let $\chi \ (\mathrm{mod}\ q)$ be a nonprincipal character, where q is a prime.

(a) Suppose that x and y are real numbers, with $y \le x < y^2$, such that $\chi(n) = 1$ for all $1 \le n \le y$. Justify the identity

$$\sum_{n\le x} \chi(n) = \psi(x, y) + \sum_{y<p\le x} \chi(p) \big\lfloor \tfrac{x}{p} \big\rfloor,$$

where $\psi(x, y)$ is defined at the beginning of Section 7.1 of Montgomery and Vaughan.

(b) Adapt the proof of Corollary 9.19, incorporating the upper bound of Theorem 9.27, to show that there exists a positive integer $m \ll_\varepsilon q^{1/(4\sqrt{e})+\varepsilon}$ such that $\chi(m) \ne 1$.